Problem 39)

a) $$E_x(\vec{r},t) = E_{x1} + E_{x2} + E_{x3} + E_{x4} = e^{i(k_0\sigma_z z - \omega t)}\Big\{E_{0x}e^{ik_0(\sigma_x x + \sigma_y y)}$$

$$- E_{0x}e^{ik_0(-\sigma_x x + \sigma_y y)} + E_{0x}e^{ik_0(\sigma_x x - \sigma_y y)} - E_{0x}e^{ik_0(-\sigma_x x - \sigma_y y)}\Big\}$$

$$= E_{0x}e^{ik_0(\sigma_z z - ct)}\left[2e^{ik_0\sigma_x x}\cos(k_0\sigma_y y) - 2e^{-ik_0\sigma_x x}\cos(k_0\sigma_y y)\right]$$

$$= 4iE_{0x}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)\,e^{i(k_0\sigma_z z - \omega t)}$$

$$= -4E_{0x}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)\sin(k_0\sigma_z z - \omega t)$$

Real part

$$E_y(\vec{r},t) = E_{oy} e^{ik_o(\sigma_z z - ct)} \left\{ e^{ik_o(\sigma_x x + \sigma_y y)} + e^{ik_o(-\sigma_x x + \sigma_y y)} - e^{ik_o(\sigma_x x - \sigma_y y)} - e^{-ik_o(\sigma_x x + \sigma_y y)} \right\}$$

$$\Rightarrow E_y(\vec{r},t) = -4E_{oy} \cos(k_o \sigma_x x) \sin(k_o \sigma_y y) \sin(k_o \sigma_z z - \omega t)$$

$$E_z(\vec{r},t) = E_{z1} + E_{z2} + E_{z3} + E_{z4} = 4E_{oz} \cos(k_o \sigma_x x) \cos(k_o \sigma_y y) \cos(k_o \sigma_z z - \omega t)$$

At $x = \pm a/2$ the tangential components of the $\vec{E}$-field are E_y, E_z. For a perfect conductor, the tangential $\vec{E}$-field must be zero; therefore,

$$\cos(\pm k_o \sigma_x a/2) = 0 \Rightarrow \cos\left(\frac{\pi a \sigma_x}{\lambda_o}\right) = 0 \Rightarrow \frac{\pi a \sigma_x}{\lambda_o} = m\pi + \frac{\pi}{2} \quad (m = \text{integer})$$

$$\Rightarrow \sigma_x = \left(m + \tfrac{1}{2}\right) \frac{\lambda_o}{a}$$

At $y = \pm b/2$ the tangential components of the $\vec{E}$-field are E_x, E_z. Therefore,

$$\cos\left(\pm k_o \sigma_y \tfrac{b}{2}\right) = 0 \Rightarrow \cos\left(\frac{\pi \sigma_y b}{\lambda_o}\right) = 0 \Rightarrow \sigma_y = \left(n + \tfrac{1}{2}\right) \frac{\lambda_o}{b} \quad n = \text{integer}$$

Any combination of m and n is acceptable so long as $\sigma_x^2 + \sigma_y^2 \le 1$; otherwise σ_z will become imaginary, and the beam will not propagate.

b) Surface charge density $\sigma_s = \epsilon_o E_\perp$.

On the walls located at $x = \pm \frac{a}{2} \Rightarrow \sigma_s(x = \pm \frac{a}{2}) = \mp \epsilon_o E_x = \pm 4\epsilon_o E_{ox} \sin(k_o \sigma_x x)$

$$\times \cos(k_o \sigma_y y) \sin(k_o \sigma_z z - \omega t) = 4\epsilon_o E_{ox} \sin\left(m\pi + \tfrac{\pi}{2}\right) \cos(k_o \sigma_y y) \sin(k_o \sigma_z z - \omega t)$$

On the walls located at $y = \pm b/2 \Rightarrow \sigma_s(y = \pm b/2) = \mp \epsilon_o E_y = \pm 4\epsilon_o E_{oy} \cos(k_o \sigma_x x)$

$$\times \sin(k_o \sigma_y y) \sin(k_o \sigma_z z - \omega t) = 4\epsilon_o E_{oy} \sin\left(n\pi + \tfrac{\pi}{2}\right) \cos(k_o \sigma_x x) \sin(k_o \sigma_z z - \omega t)$$

In order to find the surface currents we need to know the $\vec{H}$-field.

$$H_x(\vec{r},t) = H_{x1} + H_{x2} + H_{x3} + H_{x4} = -4H_{ox} \cos(k_o \sigma_x x) \sin(k_o \sigma_y y) \sin(k_o \sigma_z z - \omega t)$$

$H_y(\vec{r},t) = H_{y1}+H_{y2}+H_{y3}+H_{y4} = -4H_{oy}\sin(k_o\sigma_x x)\cos(k_o\sigma_y y)\sin(k_o\sigma_z z-\omega t)$

$H_z(\vec{r},t) = H_{z1}+H_{z2}+H_{z3}+H_{z4} = -4H_{oz}\sin(k_o\sigma_x x)\sin(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)$

Note that on the walls at $x=\pm a/2$, the perpendicular $\vec{H}$-field, H_x, is zero. Similarly, on the walls at $y=\pm b/2$, the $\perp$ field, H_y, is zero, consistent with the absence of $\vec{H}$-field from the interior regions of the metallic conductor, and with the Maxwell equation $\vec{\nabla}\cdot\vec{B}=0$.

On the walls at $x=\pm a/2 \Rightarrow \vec{J}_s(x=\pm a/2) = \mp 4H_{oz}\sin(k_o\sigma_x x)\sin(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)\hat{y}$

$\pm 4H_{oy}\sin(k_o\sigma_x x)\cos(k_o\sigma_y y)\sin(k_o\sigma_z z-\omega t)\hat{z} \Rightarrow$

$\vec{J}_s(x=\pm a/2) = 4\sin(m\pi+\frac{\pi}{2})\left\{-H_{oz}\sin(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)\hat{y}+H_{oy}\cos(k_o\sigma_y y)\sin(k_o\sigma_z z-\omega t)\hat{z}\right\}$

Similarly, on the walls at $y=\pm b/2 \Rightarrow \vec{J}_s(y=\pm b/2) = \pm 4H_{oz}\sin(k_o\sigma_x x)\sin(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)\hat{x}$

$\mp 4H_{ox}\cos(k_o\sigma_x x)\sin(k_o\sigma_y y)\sin(k_o\sigma_z z-\omega t)\hat{z} \Rightarrow$

$\vec{J}_s(y=\pm b/2) = 4\sin(n\pi+\frac{\pi}{2})\left\{H_{oz}\sin(k_o\sigma_x x)\cos(k_o\sigma_z z-\omega t)\hat{x} - H_{ox}\cos(k_o\sigma_x x)\sin(k_o\sigma_z z-\omega t)\hat{z}\right\}$

Note that at the corners the current remains continuous, flowing smoothly from one wall to the adjacent wall.

$\vec{\nabla}\cdot\vec{J}_s(x=\pm a/2) = \frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z} = 4k_o\sin(m\pi+\frac{\pi}{2})(\sigma_z H_{oy}-\sigma_y H_{oz})\cos(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)$

$= 4k_o\sin(m\pi+\frac{\pi}{2})(E_{ox}/Z_o)\cos(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)$

Also, $\frac{\partial\sigma_s(x=\pm a/2)}{\partial t} = -4\epsilon_o\omega E_{ox}\sin(m\pi+\frac{\pi}{2})\cos(k_o\sigma_y y)\cos(k_o\sigma_z z-\omega t)$

Since $k_o/Z_o = \frac{\omega/c}{\sqrt{\mu_o/\epsilon_o}} = \frac{\sqrt{\mu_o\epsilon_o}}{\sqrt{\mu_o/\epsilon_o}}\,\omega = \epsilon_o\omega$, we conclude that $\vec{\nabla}\cdot\vec{J}_s + \frac{\partial\sigma_s}{\partial t} = 0$.

The same argument can be used for the walls at $y=\pm b/2$ to prove the conservation of charge.