Problem 37)

a) Incident $\sigma_x = \sigma_y = 0$, $\sigma_z = -1$. $\vec{E}_0 = E_x \hat{x}$, $Z_0 \vec{H}_0 = \vec{\sigma} \times \vec{E}_0 = -E_x \hat{y}$

Reflected: $\sigma_x' = \sigma_y' = 0$, $\sigma_z' = +1$; $\vec{E}_0' = r E_x \hat{x}$; $Z_0 \vec{H}_0' = \vec{\sigma}' \times \vec{E}_0' = r E_x \hat{y}$

Transmitted: $\begin{cases} \sigma_x'' = \sigma_y'' = 0; \; \vec{\sigma}'' \cdot \vec{\sigma}'' = \epsilon(\omega) \Rightarrow \sigma_z''^2 = 1 - (\omega_p/\omega)^2 \Rightarrow \sigma_z'' = -i\sqrt{(\omega_p/\omega)^2 - 1} \\ \vec{E}_0'' = \tau E_x \hat{x}; \; Z_0 \vec{H}_0'' = \vec{\sigma}'' \times \vec{E}_0'' = -i\sqrt{(\omega_p/\omega)^2 - 1}\, \tau E_x \hat{y} \end{cases}$

Using the relevant $\vec{E}_0, \vec{H}_0, \vec{\sigma}$ for each beam (see above), the distributions will have the form $\vec{E}_0 \exp[ik_0(\vec{\sigma}\cdot\vec{r} - ct)] = \vec{E}_0 \exp(i\frac{\omega}{c}\sigma_z z) \exp(-i\omega t)$ and $\vec{H}_0 \exp(i\frac{\omega}{c}\sigma_z z) \exp(-i\omega t)$.

b) Continuity of $\vec{E}_\parallel$: $E_x + rE_x = \tau E_x \Rightarrow 1 + r = \tau$

Continuity of $\vec{H}_\parallel$: $-E_x + rE_x = -i\sqrt{(\omega_p/\omega)^2 - 1}\, \tau E_x \Rightarrow 1 - r = i\sqrt{(\omega_p/\omega)^2 - 1}\, \tau$

$$\Rightarrow 1-r = i\sqrt{(\omega_p/\omega)^2-1}\,(1+r) \Rightarrow r = \frac{1-i\sqrt{(\omega_p/\omega)^2-1}}{1+i\sqrt{(\omega_p/\omega)^2-1}}$$

$$\tau = 1+r = \frac{2}{1+i\sqrt{(\omega_p/\omega)^2-1}}$$

c) $$R = \left|\frac{1-i\sqrt{(\omega_p/\omega)^2-1}}{1+i\sqrt{(\omega_p/\omega)^2-1}}\right|^2 = \frac{|1-i\sqrt{\cdots}|^2}{|1+i\sqrt{\cdots}|^2} = \frac{1+(\omega_p/\omega)^2-1}{1+(\omega_p/\omega)^2-1} = 1 \quad (\text{i.e., } 100\%)$$

d) $$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}(\vec{E}''\times\vec{H}''^*) = \frac{1}{2}\mathrm{Re}\left\{\vec{E}_0'' e^{k_0\sqrt{(\omega_p/\omega)^2-1}\,z} \times \vec{H}_0''^* e^{k_0\sqrt{(\omega_p/\omega)^2-1}\,z}\right\}$$

$$= \frac{\exp(2k_0\sqrt{(\omega_p/\omega)^2-1}\,z)}{2Z_0}\mathrm{Re}\left\{\tau E_x\hat{x}\times(+i\sqrt{(\omega_p/\omega)^2-1}\,\tau^* E_x^*\hat{y})\right\} \Rightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\exp(2k_0\sqrt{(\omega_p/\omega)^2-1}\,z)}{2Z_0}|\tau|^2|E_x|^2\sqrt{(\omega_p/\omega)^2-1}\;\overset{0}{\cancel{\mathrm{Re}(i)}}\,\hat{z} = 0 \checkmark$$

This is consistent with part (c), because 100% reflectance does not leave any energy to be transmitted across the interface.

e) The $\vec{E}$- and $\vec{H}$-fields in the plasma-like medium decay exponentially as $\exp(k_0\sqrt{(\omega_p/\omega)^2-1}\,z)$. The $1/e$ point of these fields occurs at:

$$\Delta z = \frac{1}{k_0\sqrt{(\omega_p/\omega)^2-1}} = \frac{\lambda_0}{2\pi\sqrt{(\omega_p/\omega)^2-1}}$$ ← skin depth (or penetration depth)