Opti 501 **Solutions**

Problem 34)

a) Incident beam:

$$\boldsymbol{k}^{\mathrm{i}} = (\omega/c)(\sin\theta_{\mathrm{B}}\hat{\boldsymbol{x}} - \cos\theta_{\mathrm{B}}\hat{\boldsymbol{z}}), \tag{1a}$$

$$\boldsymbol{E}_{\mathrm{o}}^{\mathrm{i}} = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}}), \tag{1b}$$

$$\boldsymbol{H}_{\mathrm{o}}^{\mathrm{i}} = \boldsymbol{k}^{\mathrm{i}} \times \boldsymbol{E}_{\mathrm{o}}^{\mathrm{i}}/(\mu_{\mathrm{o}}\omega) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}, \tag{1c}$$

$$\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - z\cos\theta_{\mathrm{B}} - ct)], \tag{1d}$$

$$\boldsymbol{H}^{\mathrm{i}}(\boldsymbol{r},t) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - z\cos\theta_{\mathrm{B}} - ct)]. \tag{1e}$$

Transmitted beam:

$$\boldsymbol{k}^{\mathrm{t}} = (n\omega/c)(\sin\theta'_{\mathrm{B}}\hat{\boldsymbol{x}} - \cos\theta'_{\mathrm{B}}\hat{\boldsymbol{z}}), \tag{2a}$$

$$\boldsymbol{E}_{\mathrm{o}}^{\mathrm{t}} = E_{\mathrm{p}}^{\mathrm{t}}(\cos\theta'_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta'_{\mathrm{B}}\hat{\boldsymbol{z}}), \tag{2b}$$

$$\boldsymbol{H}_{\mathrm{o}}^{\mathrm{t}} = \boldsymbol{k}^{\mathrm{t}} \times \boldsymbol{E}_{\mathrm{o}}^{\mathrm{t}}/(\mu_{\mathrm{o}}\mu\omega) = -(nE_{\mathrm{p}}^{\mathrm{t}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}, \tag{2c}$$

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{t}}(\cos\theta'_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta'_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(xn\sin\theta'_{\mathrm{B}} - zn\cos\theta'_{\mathrm{B}} - ct)], \tag{2d}$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t) = -(nE_{\mathrm{p}}^{\mathrm{t}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(xn\sin\theta'_{\mathrm{B}} - zn\cos\theta'_{\mathrm{B}} - ct)]. \tag{2e}$$

b) At the $z=0$ interface we must have $\sin\theta_{\mathrm{B}} = n\sin\theta'_{\mathrm{B}}$ (Snell's law), so that the exponential factors will match. Also, continuity of the tangential E-field, E_x, yields $E_{\mathrm{p}}^{\mathrm{i}}\cos\theta_{\mathrm{B}} = E_{\mathrm{p}}^{\mathrm{t}}\cos\theta'_{\mathrm{B}}$, while the continuity of the tangential H-field, H_y, yields $E_{\mathrm{p}}^{\mathrm{i}} = nE_{\mathrm{p}}^{\mathrm{t}}$. Combining the last two equations, we find $n\cos\theta_{\mathrm{B}} = \cos\theta'_{\mathrm{B}}$. This equation together with Snell's law may now be solved for the two unknowns, θ_{B} and θ'_{B}, yielding $\tan\theta_{\mathrm{B}} = n$ and $\tan\theta'_{\mathrm{B}} = 1/n$. The transmitted E- and H-fields may now be written as follows:

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + n^{-2}\sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - zn^2\cos\theta_{\mathrm{B}} - ct)], \tag{3a}$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - zn^2\cos\theta_{\mathrm{B}} - ct)]. \tag{3b}$$

c) In the incidence medium, $\boldsymbol{D}^{\mathrm{i}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t)$. Therefore, at $z=0^+$ we have

$$\boldsymbol{D}^{\mathrm{i}}(x,y,z=0^+,t) = \varepsilon_{\mathrm{o}}E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - ct)]. \tag{4}$$

In the dielectric medium, however, $\boldsymbol{D}^{\mathrm{t}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}\varepsilon\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}n^2\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)$. Thus at $z=0^-$ we have

$$\boldsymbol{D}^{\mathrm{t}}(x,y,z=0^-,t) = \varepsilon_{\mathrm{o}}n^2E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + n^{-2}\sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - ct)]. \tag{5}$$

Clearly then $D_z^{\mathrm{t}}(x,y,z=0^-,t) = D_z^{\mathrm{i}}(x,y,z=0^+,t)$.

d) Incident beam: $<\boldsymbol{S}^{\mathrm{i}}(\boldsymbol{r},t)>=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\mathrm{i}*}(\boldsymbol{r},t)]=\dfrac{|E_{\mathrm{p}}^{\mathrm{i}}|^2}{2Z_{\mathrm{o}}}(\sin\theta_{\mathrm{B}}\hat{\boldsymbol{x}}-\cos\theta_{\mathrm{B}}\hat{\boldsymbol{z}}).$ (6)

Transmitted beam: $<\boldsymbol{S}^{\mathrm{t}}(\boldsymbol{r},t)>=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\mathrm{t}*}(\boldsymbol{r},t)]=\dfrac{n|E_{\mathrm{p}}^{\mathrm{t}}|^2}{2Z_{\mathrm{o}}}(\sin\theta'_{\mathrm{B}}\hat{\boldsymbol{x}}-\cos\theta'_{\mathrm{B}}\hat{\boldsymbol{z}}).$ (7)

Note that both Poynting vectors are aligned with their corresponding k-vector. However, since $E_{\mathrm{p}}^{\mathrm{i}}=nE_{\mathrm{p}}^{\mathrm{t}}$, the time-averaged Poynting vector of the incident beam is n times greater than that of the transmitted beam. Nevertheless, the cross-sectional areas of the two beams are in the ratio of $\cos\theta'_{\mathrm{B}}/\cos\theta_{\mathrm{B}}$, which is also equal to n. Therefore, the rate-of-flow of energy per unit time along the propagation direction is the same for the incident and transmitted beams, as required by energy conservation.

e) In the absence of free charge, Maxwell's 1st equation is $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=0$. Since $\boldsymbol{D}=\varepsilon_{\mathrm{o}}\boldsymbol{E}+\boldsymbol{P}$ and, by definition, $\rho_{\mathrm{bound}}^{(\mathrm{e})}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$, we have $\varepsilon_{\mathrm{o}}\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=\rho_{\mathrm{bound}}^{(\mathrm{e})}(\boldsymbol{r},t)$. Thus the discontinuity of $\varepsilon_{\mathrm{o}}E_z$ at the $z=0$ interface is equal to the bound surface-charge-density. Using Eqs.(1d) and (3a) we find

$$\sigma_s^{(\mathrm{bound})}(x,y,z=0,t)=\varepsilon_{\mathrm{o}}[E_z^{\mathrm{i}}(x,y,z=0^+,t)-E_z^{\mathrm{t}}(x,y,z=0^-,t)]$$

$$=\varepsilon_{\mathrm{o}}(1-n^{-2})E_{\mathrm{p}}^{\mathrm{i}}\sin\theta_{\mathrm{B}}\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}}-ct)]. \quad (8)$$