Problem 33) a) In problem 3 we showed that:

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{e^{-2k_o \vec{\sigma}_I \cdot \vec{r}}}{2Z_o}\left\{(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2)\vec{\sigma}_R + 2(\vec{E}_{oR}\times\vec{E}_{oI})\times\vec{\sigma}_I\right\}.$$

The first term in the above expression is proportional to $\vec{\sigma}_R$, which is orthogonal to $\vec{\sigma}_I$; thus the dot product with $\vec{\sigma}_I$ is zero. The second term is the cross-product of two vectors, one of which is $\vec{\sigma}_I$; therefore, it is also orthogonal to $\vec{\sigma}_I$. Consequently, $\langle\vec{S}\rangle\cdot\vec{\sigma}_I = 0$ ✓

b) $\vec{H}$-field energy density $= \frac{1}{2}\mu_o|\vec{H}(\vec{r},t)|^2 = \frac{1}{2}\mu_o\left|\mathrm{Re}\left\{(\vec{H}_{oR}+i\vec{H}_{oI})e^{ik_o(\vec{\sigma}\cdot\vec{r}-ct)}\right\}\right|^2$

$$= \frac{1}{2}\mu_o\left|\mathrm{Re}\left\{(\vec{H}_{oR}+i\vec{H}_{oI})\left[\cos(k_o\vec{\sigma}_R\cdot\vec{r}-\omega t)+i\sin(k_o\vec{\sigma}_R\cdot\vec{r}-\omega t)\right]\right\}\right|^2 e^{-2k_o\vec{\sigma}_I\cdot\vec{r}}$$

$$= \frac{1}{2}\mu_o\left|\vec{H}_{oR}\cos(k_o\vec{\sigma}_R\cdot\vec{r}-\omega t) - \vec{H}_{oI}\sin(k_o\vec{\sigma}_R\cdot\vec{r}-\omega t)\right|^2 \exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$$

$$= \frac{1}{2}\mu_o\left[\vec{H}_{oR}\cos(\ldots) - \vec{H}_{oI}\sin(\ldots)\right]\cdot\left[\vec{H}_{oR}\cos(\ldots) - \vec{H}_{oI}\sin(\ldots)\right]\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$$

$$= \frac{1}{2}\mu_o\left\{|\vec{H}_{oR}|^2\cos^2(\ldots) + |\vec{H}_{oI}|^2\sin^2(\ldots) - 2\vec{H}_{oR}\cdot\vec{H}_{oI}\sin(\ldots)\cos(\ldots)\right\}\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$$

$$= \frac{1}{4}\mu_o\left\{|\vec{H}_{oR}|^2 + |\vec{H}_{oI}|^2 + (|\vec{H}_{oR}|^2 - |\vec{H}_{oI}|^2)\cos(2k_o\vec{\sigma}_R\cdot\vec{r}-2\omega t) - 2\vec{H}_{oR}\cdot\vec{H}_{oI}\sin(2k_o\vec{\sigma}_R\cdot\vec{r}-2\omega t)\right\} \times \exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$$

Time-averaged magnetic-field energy density $= \frac{1}{4}\mu_o(|\vec{H}_{oR}|^2 + |\vec{H}_{oI}|^2)\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$

Now, $Z_o\vec{H}_o = \vec{\sigma}\times\vec{E}_o \Rightarrow Z_o^2\vec{H}_o\cdot\vec{H}_o^* = (\vec{\sigma}\times\vec{E}_o)\cdot(\vec{\sigma}^*\times\vec{E}_o^*) = (\vec{\sigma}\cdot\vec{\sigma}^*)(\vec{E}_o\cdot\vec{E}_o^*) - (\vec{\sigma}\cdot\vec{E}_o^*)(\vec{E}_o\cdot\vec{\sigma}^*)$

$$\Rightarrow \frac{\mu_o}{\epsilon_o}(|\vec{H}_{oR}|^2 + |\vec{H}_{oI}|^2) = (|\vec{\sigma}_R|^2 + |\vec{\sigma}_I|^2)(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2) - |(\vec{\sigma}_R + i\vec{\sigma}_I)\cdot(\vec{E}_{oR} - i\vec{E}_{oI})|^2$$

$$\Rightarrow \mu_o(|\vec{H}_{oR}|^2 + |\vec{H}_{oI}|^2) = \epsilon_o(1+2|\vec{\sigma}_I|^2)(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2) - \epsilon_o\left|\vec{\sigma}_R\cdot\vec{E}_{oR} + \vec{\sigma}_I\cdot\vec{E}_{oI} + i(\vec{\sigma}_I\cdot\vec{E}_{oR} - \vec{\sigma}_R\cdot\vec{E}_{oI})\right|^2$$

$$= \epsilon_o(1+2|\vec{\sigma}_I|^2)(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2) - \epsilon_o|2\vec{\sigma}_R\cdot\vec{E}_{oR} - 2i\vec{\sigma}_R\cdot\vec{E}_{oI}|^2 \Rightarrow$$

$\langle\vec{H}$-field energy density$\rangle = \frac{1}{4}\epsilon_o\left\{(1+2|\vec{\sigma}_I|^2)(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2) - 4(\vec{\sigma}_R\cdot\vec{E}_{oR})^2 - 4(\vec{\sigma}_R\cdot\vec{E}_{oI})^2\right\}\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})$

Not necessarily equal to $\langle\vec{E}$-field energy density$\rangle$, which is only the first term in the above expression.