Opti 501 **Solutions**

Problem 32) a) $\vec{E}$-field energy density $= \frac{1}{2}\epsilon_o \left| Re\left\{ (\vec{E}_{oR} + i\vec{E}_{oI})\left[\operatorname{Cos} k_o(\vec{\sigma}\cdot\vec{r}-ct) + i \operatorname{Sin} k_o(\vec{\sigma}\cdot\vec{r}-ct)\right]\right\}\right|^2$

$$= \frac{1}{2}\epsilon_o \left| \vec{E}_{oR}\operatorname{Cos} k_o(\vec{\sigma}\cdot\vec{r}-ct) - \vec{E}_{oI}\operatorname{Sin} k_o(\vec{\sigma}\cdot\vec{r}-ct)\right|^2 = \frac{1}{2}\epsilon_o\left(\vec{E}_{oR}\operatorname{Cos}\cdots - \vec{E}_{oI}\operatorname{Sin}\cdots\right)\cdot\left(\vec{E}_{oR}\operatorname{Cos}\cdots - \vec{E}_{oI}\operatorname{Sin}\cdots\right)$$

$$= \frac{1}{2}\epsilon_o\left\{ |\vec{E}_{oR}|^2 \operatorname{Cos}^2 k_o(\vec{\sigma}\cdot\vec{r}-ct) + |\vec{E}_{oI}|^2 \operatorname{Sin}^2 k_o(\vec{\sigma}\cdot\vec{r}-ct) - 2\vec{E}_{oR}\cdot\vec{E}_{oI}\operatorname{Sin} k_o(\vec{\sigma}\cdot\vec{r}-ct)\operatorname{Cos} k_o(\vec{\sigma}\cdot\vec{r}-ct)\right\}$$

$$= \frac{1}{2}\epsilon_o\left\{ \frac{1}{2}\left(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2\right) + \frac{1}{2}\left(|\vec{E}_{oR}|^2 - |\vec{E}_{oI}|^2\right)\operatorname{Cos} 2k_o(\vec{\sigma}\cdot\vec{r}-ct) - \vec{E}_{oR}\cdot\vec{E}_{oI}\operatorname{Sin} 2k_o(\vec{\sigma}\cdot\vec{r}-ct)\right\}$$

$\Rightarrow$ Time-averaged $\vec{E}$-field energy density $= \frac{1}{4}\epsilon_o\left(|\vec{E}_{oR}|^2 + |\vec{E}_{oI}|^2\right)$

b) Similarly,

Time-averaged $\vec{H}$-field energy density $= \frac{1}{4}\mu_o\left(|\vec{H}_{oR}|^2 + |\vec{H}_{oI}|^2\right)$

c) $$Z_o\vec{H}_o = \vec{\sigma}\times\vec{E}_o \Rightarrow Z_o(\vec{H}_{oR} + i\vec{H}_{oI}) = \vec{\sigma}_R \times (\vec{E}_{oR} + i\vec{E}_{oI}) \Rightarrow \begin{cases} Z_o\vec{H}_{oR} = \vec{\sigma}_R\times\vec{E}_{oR} \\ Z_o\vec{H}_{oI} = \vec{\sigma}_R\times\vec{E}_{oI}\end{cases}$$

However, $\vec{\sigma}\cdot\vec{\sigma}=1 \Rightarrow \vec{\sigma}_R\cdot\vec{\sigma}_R=1$ for a homogeneous plane-wave. Therefore, $\vec{\sigma}_R$ is a vector of unit length. Moreover, $\vec{\sigma}\cdot\vec{E}_0=0 \Rightarrow \vec{\sigma}_R\cdot\vec{E}_{0R}=0$ and $\vec{\sigma}_R\cdot\vec{E}_{0I}=0$; that is, $\vec{\sigma}_R$ is $\perp$ to both $\vec{E}_{0R}$ and $\vec{E}_{0I}$. From all these considerations, we conclude that $Z_0|\vec{H}_{0R}|=|\vec{E}_{0R}|$ and $Z_0|\vec{H}_{0I}|=|\vec{E}_{0I}|$.

Time-averaged $\vec{H}$-field energy density $=\frac{1}{4}\mu_0\left(\frac{1}{Z_0^2}|\vec{E}_{0R}|^2+\frac{1}{Z_0^2}|\vec{E}_{0I}|^2\right)$

$=\frac{1}{4}\mu_0\frac{1}{\mu_0/\epsilon_0}\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)=\frac{1}{4}\epsilon_0\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)=$ Time-averaged $\vec{E}$-field energy density ✓

d) $\langle \vec{S}\rangle=\frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)=\frac{1}{2}\mathrm{Re}(\vec{E}_0\times\vec{H}_0^*)=\frac{1}{2}\mathrm{Re}\left\{(\vec{E}_{0R}+i\vec{E}_{0I})\times(\vec{H}_{0R}-i\vec{H}_{0I})\right\}$

$=\frac{1}{2}\vec{E}_{0R}\times\vec{H}_{0R}+\frac{1}{2}\vec{E}_{0I}\times\vec{H}_{0I}=\frac{1}{2Z_0}\left[\vec{E}_{0R}\times(\vec{\sigma}_R\times\vec{E}_{0R})+\vec{E}_{0I}\times(\vec{\sigma}_R\times\vec{E}_{0I})\right]$

$=\frac{1}{2Z_0}\left\{(\vec{E}_{0R}\cdot\vec{E}_{0R})\vec{\sigma}_R-\underbrace{\cancel{(\vec{E}_{0R}\cdot\vec{\sigma}_R)}}_{0}\vec{E}_{0R}+(\vec{E}_{0I}\cdot\vec{E}_{0I})\vec{\sigma}_R-\underbrace{\cancel{(\vec{E}_{0I}\cdot\vec{\sigma}_R)}}_{0}\vec{E}_0\right\}\Rightarrow$

$\langle\vec{S}(\vec{r},t)\rangle=\frac{1}{2Z_0}\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)\vec{\sigma}_R$

Thus the electromagnetic energy passing through unit area in unit time along the propagation direction $\vec{\sigma}_R$ is $\frac{1}{2Z_0}\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)$.

e) Since electromagnetic fields travel with speed $c=1/\sqrt{\mu_0\epsilon_0}$ in free-space, the total $\vec{E}$-field energy + $\vec{H}$-field energy contained in a volume of cross-sectional area = unity and length = c must be delivered to this volume in unit time. Thus:

$c\left\{\frac{1}{4}\epsilon_0(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2)+\frac{1}{4}\mu_0(|\vec{H}_{0R}|^2+|\vec{H}_{0I}|^2)\right\}=\frac{\epsilon_0}{2\sqrt{\mu_0\epsilon_0}}\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)$

$=\frac{1}{2Z_0}\left(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2\right)=\langle S\rangle$ ✓