Opti 501 **Solutions**

Problem 31)

$$\langle \vec{S}(\vec{r},t)\rangle = \tfrac{1}{2}\mathrm{Re}(E\times H^*) = \tfrac{1}{2}\mathrm{Re}\left\{\vec{E}_0\, e^{ik_0(\vec{\sigma}\cdot\vec{r}-ct)} \times H_0^*\, e^{-ik_0(\vec{\sigma}^*\cdot\vec{r}-ct)}\right\}$$

$$= \tfrac{1}{2}\mathrm{Re}(\vec{E}_0\times\vec{H}_0^*)\, e^{-2k_0\vec{\sigma}_I\cdot\vec{r}} = \frac{1}{2Z_0}\mathrm{Re}\{\vec{E}_0\times(\vec{\sigma}^*\times\vec{E}_0^*)\}\, e^{-2k_0\vec{\sigma}_I\cdot\vec{r}}$$

Since $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$, we may write $\vec{E}_0\times(\vec{\sigma}^*\times\vec{E}_0^*) = (\vec{E}_0\cdot\vec{E}_0^*)\vec{\sigma}^*$

$$-(\vec{E}_0\cdot\vec{\sigma}^*)\vec{E}_0^* = \left[(\vec{E}_{0R}+i\vec{E}_{0I})\cdot(\vec{E}_{0R}-i\vec{E}_{0I})\right](\vec{\sigma}_R - i\vec{\sigma}_I) - \left[(\vec{E}_{0R}+i\vec{E}_{0I})\cdot(\vec{\sigma}_R - i\vec{\sigma}_I)\right](\vec{E}_{0R}-i\vec{E}_{0I})$$

$$=(\vec{E}_{0R}\cdot\vec{E}_{0R} + \vec{E}_{0I}\cdot\vec{E}_{0I})(\vec{\sigma}_R - i\vec{\sigma}_I) - \left[(\vec{E}_{0R}\cdot\vec{\sigma}_R + \vec{E}_{0I}\cdot\vec{\sigma}_I) + i(\vec{E}_{0I}\cdot\vec{\sigma}_R - \vec{E}_{0R}\cdot\vec{\sigma}_I)\right](\vec{E}_{0R} - i\vec{E}_{0I})$$

We saw in Prob. 2 above that $\vec{\sigma}\cdot\vec{E}_0 = 0$ implies that $\vec{\sigma}_R\cdot\vec{E}_{0R} = \vec{\sigma}_I\cdot\vec{E}_{0I}$ and $\vec{\sigma}_R\cdot\vec{E}_{0I} = -\vec{\sigma}_I\cdot\vec{E}_{0R}$. Therefore,

$$\vec{E}_0\times(\vec{\sigma}^*\times\vec{E}_0^*) = (|\vec{E}_{0R}|^2 + |\vec{E}_{0I}|^2)(\vec{\sigma}_R - i\vec{\sigma}_I) - 2(\vec{E}_{0I}\cdot\vec{\sigma}_I - i\vec{E}_{0R}\cdot\vec{\sigma}_I)(\vec{E}_{0R} - i\vec{E}_{0I})$$

Thus $\mathrm{Re}\{\vec{E}_0\times(\vec{\sigma}^*\times\vec{E}_0^*)\} = (|\vec{E}_{0R}|^2 + |\vec{E}_{0I}|^2)\vec{\sigma}_R - 2(\vec{E}_{0I}\cdot\vec{\sigma}_I)\vec{E}_{0R} + 2(\vec{E}_{0R}\cdot\vec{\sigma}_I)\vec{E}_{0I}$

$$= (|\vec{E}_{0R}|^2 + |\vec{E}_{0I}|^2)\vec{\sigma}_R - 2\vec{\sigma}_I\times(\vec{E}_{0R}\times\vec{E}_{0I}).$$

Consequently,

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{e^{-2k_0\vec{\sigma}_I\cdot\vec{r}}}{2Z_0}\left\{(|\vec{E}_{0R}|^2 + |\vec{E}_{0I}|^2)\vec{\sigma}_R + 2(\vec{E}_{0R}\times\vec{E}_{0I})\times\vec{\sigma}_I\right\}$$

For a homogeneous plane-wave $\vec{\sigma}_I = 0$; therefore,

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{|\vec{E}_{0R}|^2 + |\vec{E}_{0I}|^2}{2Z_0}\vec{\sigma}_R$$