Opti 501 **Solutions**

Problem 27)

Incident beam: $\sigma_x = \sigma_y = 0$, $\sigma_z = -1$, $\vec{E}_0 = E_x \hat{x}$, $Z_0 \vec{H}_0 = \vec{\sigma} \times \vec{E}_0 = -E_x \hat{y}$.

Reflected beam: $\sigma_x' = \sigma_y' = 0$, $\sigma_z' = +1$, $\vec{E}_0' = rE_x \hat{x}$, $Z_0 \vec{H}_0' = \vec{\sigma}' \times \vec{E}_0' = rE_x \hat{y}$

First beam in the slab: $\sigma_{x1} = \sigma_{y1} = 0$, $\sigma_{z1} = -n_0$, $\vec{E}_{01} = A\hat{x}$, $Z_0 \vec{H}_{01} = \vec{\sigma}_1 \times \vec{E}_{01} = -n_0 A \hat{y}$

Second beam in the slab: $\sigma_{x2} = \sigma_{y2} = 0$, $\sigma_{z2} = +n_0$, $\vec{E}_{02} = B\hat{x}$, $Z_0 \vec{H}_{02} = \vec{\sigma}_2 \times \vec{E}_{02} = n_0 B \hat{y}$

Transmitted beam: $\sigma_x'' = \sigma_y'' = 0$, $\sigma_z'' = -1$, $\vec{E}_0'' = \tau E_x \hat{x}$, $Z_0 \vec{H}_0'' = \vec{\sigma}'' \times \vec{E}_0'' = -\tau E_x \hat{y}$

Recall that the general expression for the fields is $\vec{E}(\vec{r},t) = \vec{E}_0 \exp[ik_0(\vec{\sigma}\cdot\vec{r} - ct)]$.

Matching the boundary conditions at $z=0$:

$$\begin{cases} E_x + rE_x = A + B \Rightarrow (1+r)E_x = A+B \\ H_y + H_y' = H_{1y} + H_{2y} \Rightarrow -E_x + rE_x = -n_0 A + n_0 B \Rightarrow \frac{1}{n_0}(1-r)E_x = A - B \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{1}{2}(1+r)E_x + \frac{1}{2n_0}(1-r)E_x = \frac{1}{2}\left[(1+\frac{1}{n_0}) + (1-\frac{1}{n_0})r\right]E_x = \frac{1}{2n_0}\left[(n_0+1)+(n_0-1)r\right]E_x \\ B = \frac{1}{2}(1+r)E_x - \frac{1}{2n_0}(1-r)E_x = \frac{1}{2}\left[(1-\frac{1}{n_0}) + (1+\frac{1}{n_0})r\right]E_x = \frac{1}{2n_0}\left[(n_0-1)+(n_0+1)r\right]E_x \end{cases}$$

Matching the boundary conditions at $z=-d$:

$$\begin{cases} E_x: \; Ae^{+ik_0 n_0 d} + Be^{-ik_0 n_0 d} = \tau E_x e^{+ik_0 d} \\ H_y: \; -n_0 A e^{+ik_0 n_0 d} + n_0 B e^{-ik_0 n_0 d} = -\tau E_x e^{+ik_0 d} \end{cases} \Rightarrow \text{(using } A \text{ \& } B \text{ found above)}$$

$$\begin{cases} \frac{1}{2n_0}\left[(n_0+1)+(n_0-1)r\right]e^{2ik_0 n_0 d} + \frac{1}{2n_0}\left[(n_0-1)+(n_0+1)r\right] = e^{ik_0(n_0+1)d}\,\tau \\ \frac{1}{2}\left[(n_0+1)+(n_0-1)r\right]e^{2ik_0 n_0 d} - \frac{1}{2}\left[(n_0-1)+(n_0+1)r\right] = e^{ik_0(n_0+1)d}\,\tau \end{cases}$$

Next we set the left-hand-sides of the above equations equal to each other:

$$\left(\frac{1}{n_0}-1\right)\left[(n_0+1)+(n_0-1)r\right]e^{2ik_0 n_0 d} + \left(\frac{1}{n_0}+1\right)\left[(n_0-1)+(n_0+1)r\right] = 0 \quad \Rightarrow$$

$$(1-n_0)(n_0+1)\left\{1 + \frac{n_0-1}{n_0+1}r\right\}e^{2ik_0 n_0 d} + (n_0+1)^2\left[\frac{n_0-1}{n_0+1} + r\right] = 0 \quad \Rightarrow$$

$$-\left(\frac{n_o-1}{n_o+1}\right)\left[1+\left(\frac{n_o-1}{n_o+1}\right)r\right]e^{i4\pi n_o d/\lambda_o}+\left(\frac{n_o-1}{n_o+1}\right)+r=0 \Rightarrow$$

$$r\left[1-\left(\frac{n_o-1}{n_o+1}\right)^2 e^{i4\pi n_o d/\lambda_o}\right]=\left(\frac{n_o-1}{n_o+1}\right)\left(e^{i4\pi n_o d/\lambda_o}-1\right) \Rightarrow$$

$$r=\frac{\left(\frac{n_o-1}{n_o+1}\right)\left(e^{i4\pi n_o d/\lambda_o}-1\right)}{1-\left(\frac{n_o-1}{n_o+1}\right)^2 e^{i4\pi n_o d/\lambda_o}}$$

$\rightarrow$ To determine A, B, and τ, simply substitute for r in the preceding equations.

When $d=\lambda_o/2n_o$ (or any integer-multiple of $\lambda_o/2n_o$), we'll have $e^{i4\pi n_o d/\lambda_o}=e^{i2\pi}=1$, which makes the numerator of r equal to zero. ✓

To simplify the notation, we define the single-surface reflection coefficient $\rho=(n_o-1)/(n_o+1)$, and the (double-path) phase factor $\exp(i\phi)=\exp(i4\pi n_o d/\lambda_o)$. We'll have:

$$R=|r|^2=\left|\frac{\rho(e^{i\phi}-1)}{1-\rho^2 e^{i\phi}}\right|^2=\rho^2\frac{(e^{i\phi}-1)(e^{-i\phi}-1)}{(1-\rho^2 e^{i\phi})(1-\rho^2 e^{-i\phi})} \Rightarrow$$

$$R=\rho^2\frac{1+1-2\cos\phi}{1+\rho^4-2\rho^2\cos\phi}=2\rho^2\frac{(1-\cos\phi)}{(1-\rho^2)^2+2\rho^2(1-\cos\phi)}=\frac{4\rho^2\sin^2\phi/2}{(1-\rho^2)^2+4\rho^2\sin^2\frac{\phi}{2}}$$

$$\Rightarrow R=\frac{1}{1+\left(\frac{1-\rho^2}{2\rho\sin\frac{\phi}{2}}\right)^2}$$

To maximize the reflectivity R, the denominator must be a minimum. Since ρ is fixed, $\sin^2\frac{\phi}{2}$ must be maximized, which happens when $\phi=\pi, 3\pi, 5\pi$, etc. Thus the optimum values of d for maximum reflectance are $d=\frac{\lambda_o}{4n_o}, \frac{3\lambda_o}{4n_o}, \frac{5\lambda_o}{4n_o}$, etc.

At optimum thickness, $\sin^2\phi/2=1$, resulting in:

$$R_{max}=\frac{4\rho^2}{(1+\rho^2)^2}=\left(\frac{2\rho}{1+\rho^2}\right)^2=\left(\frac{n_o^2-1}{n_o^2+1}\right)^2$$