**Opti 501** **Solutions**

**Problem 24)**

a) $\sigma_x = \sigma_x' = \sigma_x'' \Rightarrow \sigma_x' = \sigma_x'' = n_o \sin\theta$

$\sigma_y = \sigma_y' = \sigma_y'' = 0$

$\vec{\sigma}' \cdot \vec{\sigma}' = n_o^2 \Rightarrow \sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2 = n_o^2 \Rightarrow \sigma_z' = +n_o \cos\theta$

$\vec{\sigma}'' \cdot \vec{\sigma}'' = \epsilon(\omega) \Rightarrow \sigma_x''^2 + \sigma_y''^2 + \sigma_z''^2 = \epsilon(\omega) \Rightarrow \sigma_z'' = -\sqrt{\epsilon(\omega) - n_o^2 \sin^2\theta}$

Therefore, $\vec{\sigma}' = n_o \sin\theta\, \hat{x} + n_o \cos\theta\, \hat{z}$ and $\vec{\sigma}'' = n_o \sin\theta\, \hat{x} - \sqrt{\epsilon(\omega) - n_o^2 \sin^2\theta}\, \hat{z}$

b) $\vec{\sigma} \cdot \vec{E}_o = 0 \Rightarrow \sigma_x E_x + \sigma_y E_y + \sigma_z E_z = 0 \Rightarrow n_o \sin\theta\, E_x - n_o \cos\theta\, E_z = 0 \Rightarrow \frac{E_z}{E_x} = \tan\theta$

$\vec{\sigma}' \cdot \vec{E}_o' = 0 \Rightarrow \sigma_x' E_x' + \sigma_y' E_y' + \sigma_z' E_z' = 0 \Rightarrow n_o \sin\theta\, E_x' + n_o \cos\theta\, E_z' = 0 \Rightarrow \frac{E_z'}{E_x'} = -\tan\theta$

$\vec{\sigma}'' \cdot \vec{E}_o'' = 0 \Rightarrow n_o \sin\theta\, E_x'' - \sqrt{\epsilon(\omega) - n_o^2 \sin^2\theta}\; E_z'' = 0 \Rightarrow \frac{E_z''}{E_x''} = \frac{n_o \sin\theta}{\sqrt{\epsilon(\omega) - n_o^2 \sin^2\theta}}$

c) $Z_o \vec{H}_o = \vec{\sigma} \times \vec{E}_o = (n_o \sin\theta\, \hat{x} - n_o \cos\theta\, \hat{z}) \times (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) = n_o \cos\theta\, E_y\, \hat{x} -$
$(n_o \sin\theta\, E_z + n_o \cos\theta\, E_x)\hat{y} + n_o \sin\theta\, E_y\, \hat{z} = n_o \cos\theta\, E_y\, \hat{x} - n_o(\sin\theta \tan\theta + \cos\theta) E_x\, \hat{y} + n_o \sin\theta\, E_y\, \hat{z}$

$\Rightarrow Z_o \vec{H}_o = n_o \cos\theta\, E_y\, \hat{x} - \frac{n_o}{\cos\theta} E_x\, \hat{y} + n_o \sin\theta\, E_y\, \hat{z}$

$$Z_0 \vec{H}_0' = \vec{\sigma}' \times \vec{E}_0' = -n_0 \cos\theta\, E_y' \hat{x} + \frac{n_0}{\cos\theta} E_x' \hat{y} + n_0 \sin\theta\, E_y' \hat{z}$$

$$Z_0 \vec{H}_0'' = \vec{\sigma}'' \times \vec{E}_0'' = (n_0 \sin\theta\, \hat{x} - \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}\, \hat{z}) \times (E_x'' \hat{x} + E_y'' \hat{y} + E_z'' \hat{z}) =$$

$$\sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}\, E_y'' \hat{x} - \frac{\epsilon(\omega)}{\sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}} E_x'' \hat{y} + n_0 \sin\theta\, E_y'' \hat{z}$$

d) $\begin{cases} E_x + E_x' = E_x'' \\ H_y + H_y' = H_y'' \Rightarrow -\dfrac{n_0}{\cos\theta} E_x + \dfrac{n_0}{\cos\theta} E_x' = -\dfrac{\epsilon(\omega)}{\sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}} (E_x + E_x') \end{cases}$

$$\Rightarrow \left( \frac{n_0}{\cos\theta} + \frac{\epsilon(\omega)}{\sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}} \right) E_x' = \left( \frac{n_0}{\cos\theta} - \frac{\epsilon(\omega)}{\sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}} \right) E_x \Rightarrow$$

$$r_p = \frac{E_x'}{E_x} = \frac{n_0 \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta} - \epsilon(\omega) \cos\theta}{n_0 \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta} + \epsilon(\omega) \cos\theta}$$

$\begin{cases} E_y + E_y' = E_y'' \\ H_x + H_x' = H_x'' \Rightarrow n_0 \cos\theta\, E_y - n_0 \cos\theta\, E_y' = \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}\, (E_y + E_y') \Rightarrow \end{cases}$

$$\left( n_0 \cos\theta - \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta} \right) E_y = \left( n_0 \cos\theta + \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta} \right) E_y' \Rightarrow$$

$$r_s = \frac{E_y'}{E_y} = \frac{n_0 \cos\theta - \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}}{n_0 \cos\theta + \sqrt{\epsilon(\omega) - n_0^2 \sin^2\theta}}$$

e) $\vec{\sigma}'' = n_0 \sin\theta\, \hat{x} - \sqrt{(\epsilon_R(\omega) - n_0^2 \sin^2\theta) + i\epsilon_I(\omega)}\, \hat{z} = \sigma_{xR}'' \hat{x} - (\sigma_{zR}'' + i\sigma_{zI}'') \hat{z}$

Obviously $\sigma_x''$ is real (written as $\sigma_{xR}''$, for emphasis), $\sigma_y'' = 0$, and $\sigma_z''$ is the only component of $\vec{\sigma}''$ that can possibly have an imaginary part. For this reason, we have written $\sigma_z''$ explicitly as $\sigma_{zR}'' + i\sigma_{zI}''$. It is also clear that $\sigma_{zI}'' \neq 0$ if $\epsilon_I(\omega) \neq 0$ or $n_0^2 \sin^2\theta > \epsilon_R(\omega)$. In the first instance, the medium into which the light is being transmitted is absorptive; in the second instance, light is being reflected by total internal reflection at the interface. Either way, the light amplitude in the second medium decays exponentially away from the interface, as follows:

$$\vec{E}(\vec{r},t) = \vec{E}_0'' e^{ik_0(\vec{\sigma}''\cdot\vec{r}-ct)} = \vec{E}_0'' e^{k_0\sigma''_{zI} z}\, e^{ik_0(\sigma''_{xR} x - \sigma''_{zR} z - ct)}$$

The exponential decay-factor is thus $\exp(k_0 \sigma''_{zI} z) = \exp(2\pi\sigma''_{zI} z/\lambda_0)$

The "penetration depth" may be defined as the 1/e point of this exponential function, that is, $\Delta z = \lambda_0/(2\pi\sigma''_{zI})$.

f) We use the formula for $\langle \vec{S}(\vec{r},t)\rangle$ derived in Problem 3 above.

Here $\vec{\sigma}_R'' = n_0 \sin\theta\,\hat{x} - \mathrm{Re}\sqrt{(\epsilon_R(\omega)-n_0^2\sin^2\theta)+i\,\epsilon_I(\omega)}\;\hat{z}$,

$\vec{\sigma}_I'' = -\,\mathrm{Im}\sqrt{(\epsilon_R(\omega)-n_0^2\sin^2\theta)+i\,\epsilon_I(\omega)}\;\hat{z}$, and $\vec{E}_0'' = E_x''\hat{x} + E_y''\hat{y} + E_z''\hat{z}$, where

$$\begin{cases} E_x'' = E_x + E_x' = (1+r_p)\,E_x \\ E_y'' = E_y + E_y' = (1+r_s)\,E_y \\ E_z'' = \dfrac{n_0 \sin\theta}{\sqrt{(\epsilon_R(\omega)-n_0^2\sin^2\theta)+i\,\epsilon_I(\omega)}}\,(1+r_p)\,E_x \end{cases}$$

$\leftarrow$ (See part (b) for the ratio $E_z''/E_x''$)

One must then separate the real and imaginary parts of $E_x''$, $E_y''$, $E_z''$ to form $\vec{E}_{0R}''$ and $\vec{E}_{0I}''$, then replace these in the expression for $\langle\vec{S}(\vec{r},t)\rangle$ derived in Problem 3 above. The procedure is straightforward but tedious.

For conservation of power, we need only consider the $\hat{z}$-component of $\langle\vec{S}(\vec{r},t)\rangle$. In the second medium, $\vec{\sigma}_I''$ is along the $\hat{z}$-axis; therefore, the term $(\vec{E}_{0R}''\times\vec{E}_{0I}'')\times\vec{\sigma}_I''$

is parallel to the interface and need not be considered any further. Also, the power entering the second medium must be evaluated at $z=0$; therefore, $\exp(-2k_0\vec{\sigma}_I''\cdot\vec{r}) = \exp(-2k_0\sigma_{zI}''z) = 1$ at $z=0$. The remaining term, $(|\vec{E}_{oR}''|^2+|\vec{E}_{oI}''|^2)\vec{\sigma}_R''$, may be written $(|E_x''|^2+|E_y''|^2+|E_z''|^2)(\sigma_{xR}''\hat{x}+\sigma_{zR}''\hat{z})$.

Therefore,

$$\langle S_z''(\vec{r},t)\rangle\Big|_{z=0} = \frac{1}{2Z_0}\left(|E_x''|^2+|E_y''|^2+|E_z''|^2\right)\sigma_{zR}'' \quad \leftarrow \text{Transmitted beam}$$

$$= -\frac{1}{2Z_0}\left(|E_x+E_x'|^2+|E_y+E_y'|^2+|E_z''|^2\right)\mathrm{Re}\left(\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta}\right)$$

$$= -\frac{1}{2Z_0}\left\{|1+r_p|^2|E_x|^2+|1+r_s|^2|E_y|^2+\frac{n_0^2\sin^2\theta}{|\epsilon(\omega)-n_0^2\sin^2\theta|}|1+r_p|^2|E_x|^2\right\}\mathrm{Re}\left(\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta}\right)$$

$$= -\frac{1}{2Z_0}\left\{\left[1+\frac{n_0^2\sin^2\theta}{|\epsilon(\omega)-n_0^2\sin^2\theta|}\right]|1+r_p|^2|E_x|^2+|1+r_s|^2|E_y|^2\right\}\mathrm{Re}\left(\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta}\right)$$

By the same token we can write, for the incident and reflected waves,

$$\langle S_z(\vec{r},t)\rangle\Big|_{z=0} = -\frac{1}{2Z_0}\left\{(1+\tan^2\theta)|E_x|^2+|E_y|^2\right\}n_0\cos\theta \quad \leftarrow \text{Incident beam}$$

$$\langle S_z'(\vec{r},t)\rangle\Big|_{z=0} = +\frac{1}{2Z_0}\left\{(1+\tan^2\theta)|r_p|^2|E_x|^2+|r_s|^2|E_y|^2\right\}n_0\cos\theta \quad \leftarrow \text{Reflected beam}$$

We may thus verify the conservation of power for p- and s-polarized components of the beam separately, by confirming the following equalities:

$$\begin{cases}(1+\tan^2\theta)(1-|r_p|^2)n_0\cos\theta = \left(1+\dfrac{n_0^2\sin^2\theta}{|\epsilon(\omega)-n_0^2\sin^2\theta|}\right)|1+r_p|^2\,\mathrm{Re}\left(\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta}\right) & \leftarrow \text{p-polarization}\\[2ex] (1-|r_s|^2)n_0\cos\theta = |1+r_s|^2\,\mathrm{Re}\left(\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta}\right) & \leftarrow \text{s-polarization}\end{cases}$$

In what follows, we prove the above identities, first for s-polarized light and then for p-polarized light.

I) The Case of S-polarized light:

$$\left(1-\frac{\left|n_0\cos\theta-\sqrt{\epsilon-n_0^2\sin^2\theta}\right|^2}{\left|n_0\cos\theta+\sqrt{\epsilon-n_0^2\sin^2\theta}\right|^2}\right)n_0\cos\theta=\left|1+\frac{n_0\cos\theta-\sqrt{\epsilon-n_0^2\sin^2\theta}}{n_0\cos\theta+\sqrt{\epsilon-n_0^2\sin^2\theta}}\right|^2\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}$$

$$\Rightarrow\left\{\left|n_0\cos\theta+\sqrt{\epsilon-n_0^2\sin^2\theta}\right|^2-\left|n_0\cos\theta-\sqrt{\epsilon-n_0^2\sin^2\theta}\right|^2\right\}n_0\cos\theta=(2n_0\cos\theta)^2\,\mathrm{Re}\left(\sqrt{\epsilon-n_0^2\sin^2\theta}\right)$$

$$\Rightarrow\left(4n_0\cos\theta\,\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}\right)n_0\cos\theta=4n_0^2\cos^2\theta\,\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}\quad\checkmark$$

II) The Case of P-polarized light:

$$\left(1-\frac{\left|n_0\sqrt{\epsilon-n_0^2\sin^2\theta}-\epsilon\cos\theta\right|^2}{\left|n_0\sqrt{\epsilon-n_0^2\sin^2\theta}+\epsilon\cos\theta\right|^2}\right)\frac{n_0}{\cos\theta}=\left(1+\frac{n_0^2\sin^2\theta}{|\epsilon-n_0^2\sin^2\theta|}\right)\left|1+\frac{n_0\sqrt{\epsilon-n_0^2\sin^2\theta}-\epsilon\cos\theta}{n_0\sqrt{\epsilon-n_0^2\sin^2\theta}+\epsilon\cos\theta}\right|^2\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}$$

$$\Rightarrow\left\{\left|n_0\sqrt{\epsilon-n_0^2\sin^2\theta}+\epsilon\cos\theta\right|^2-\left|n_0\sqrt{\epsilon-n_0^2\sin^2\theta}-\epsilon\cos\theta\right|^2\right\}\frac{n_0}{\cos\theta}=\left(1+\frac{n_0^2\sin^2\theta}{|\epsilon-n_0^2\sin^2\theta|}\right)\left|2n_0\sqrt{\epsilon-n_0^2\sin^2\theta}\right|^2\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}$$

$$\Rightarrow 4\cancel{n_0}\cancel{\cos\theta}\,\mathrm{Re}\left(\epsilon^*\sqrt{\epsilon-n_0^2\sin^2\theta}\right)\frac{\cancel{n_0}}{\cancel{\cos\theta}}=4\cancel{n_0^2}\left(|\epsilon-n_0^2\sin^2\theta|+n_0^2\sin^2\theta\right)\mathrm{Re}\left(\sqrt{\epsilon-n_0^2\sin^2\theta}\right)$$

$$\Rightarrow\mathrm{Re}\left\{(\epsilon_R-i\epsilon_I)\left[\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}+i\,\mathrm{Im}\sqrt{\epsilon-n_0^2\sin^2\theta}\right]\right\}=\left(|\epsilon-n_0^2\sin^2\theta|+n_0^2\sin^2\theta\right)\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}$$

$$\Rightarrow\epsilon_R+\epsilon_I\frac{\mathrm{Im}\sqrt{\epsilon-n_0^2\sin^2\theta}}{\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}}=|\epsilon-n_0^2\sin^2\theta|+n_0^2\sin^2\theta$$

$$\Rightarrow\frac{\mathrm{Im}\sqrt{\epsilon-n_0^2\sin^2\theta}}{\mathrm{Re}\sqrt{\epsilon-n_0^2\sin^2\theta}}=\frac{|\epsilon-n_0^2\sin^2\theta|-(\epsilon_R-n_0^2\sin^2\theta)}{\epsilon_I}$$

Let $\epsilon-n_0^2\sin^2\theta=(\epsilon_R-n_0^2\sin^2\theta)+i\epsilon_I=Ae^{i\phi}$. The above equation can then be written as follows:

$$\frac{\sqrt{A}\sin\phi/2}{\sqrt{A}\cos\phi/2}=\frac{A-A\cos\phi}{A\sin\phi}\Rightarrow\tan\frac{\phi}{2}=\frac{1-\cos\phi}{\sin\phi}=\frac{2\sin^2\phi/2}{2\sin\phi/2\cos\phi/2}\Rightarrow\tan\frac{\phi}{2}=\tan\frac{\phi}{2}\ \checkmark$$

g) $r_p(\theta=\theta_B)=0\Rightarrow n_0\sqrt{\epsilon(\omega)-n_0^2\sin^2\theta_B}=\epsilon(\omega)\cos\theta_B\Rightarrow n_0^2(n^2-n_0^2\sin^2\theta_B)=n^4\cos^2\theta_B$

$\Rightarrow n_0^2\left(\frac{n^2}{\cos^2\theta_B}-n_0^2\tan^2\theta_B\right)=n^4\Rightarrow n^2(1+\tan^2\theta_B)-n_0^2\tan^2\theta_B=n^4/n_0^2\Rightarrow\tan\theta_B=n/n_0$