Opti 501 **Solutions**

Problem 23)

$$\langle \vec{S}(\vec{r},t)\rangle = \tfrac{1}{2}\,\mathrm{Re}\,[\vec{E}(\vec{r},t)\times\vec{H}^*(\vec{r},t)] = \tfrac{1}{2}\,\mathrm{Re}\left\{\vec{E}_o\times\vec{H}_o^*\,e^{ik_o\vec{\sigma}\cdot\vec{r}}\,e^{-ik_o\vec{\sigma}^*\cdot\vec{r}}\right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\,(\vec{E}_o\times\vec{H}_o^*)\,e^{-2k_o\vec{\sigma}_I\cdot\vec{r}}$$

Now, $Z_o\vec{H}_o = \vec{\sigma}\times\vec{E}_o \Rightarrow \vec{H}_o^* = \frac{1}{Z_o}\vec{\sigma}^*\times\vec{E}_o^*$. Therefore:

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2Z_o}\,\mathrm{Re}\left\{\vec{E}_o\times(\vec{\sigma}^*\times\vec{E}_o^*)\right\}e^{-2k_o\vec{\sigma}_I\cdot\vec{r}}$$

$$= \frac{\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})}{2Z_o}\,\mathrm{Re}\left[(\vec{E}_o\cdot\vec{E}_o^*)\vec{\sigma}^* - (\vec{E}_o\cdot\vec{\sigma}^*)\vec{E}_o^*\right]$$

Now, $(\vec{E}_o\cdot\vec{E}_o^*)\vec{\sigma}^* = \left[(\vec{E}_{oR}+i\vec{E}_{oI})\cdot(\vec{E}_{oR}-i\vec{E}_{oI})\right](\vec{\sigma}_R-i\vec{\sigma}_I) = \left(|\vec{E}_{oR}|^2+|\vec{E}_{oI}|^2\right)(\vec{\sigma}_R-i\vec{\sigma}_I)$

$\Rightarrow \mathrm{Re}\,[(\vec{E}_o\cdot\vec{E}_o^*)\vec{\sigma}^*] = (|\vec{E}_{oR}|^2+|\vec{E}_{oI}|^2)\,\vec{\sigma}_R$.

Similarly, $(\vec{E}_o\cdot\vec{\sigma}^*)\vec{E}_o^* = \left[(\vec{E}_{oR}+i\vec{E}_{oI})\cdot(\vec{\sigma}_R-i\vec{\sigma}_I)\right](\vec{E}_{oR}-i\vec{E}_{oI})$

$= \left[(\vec{E}_{oR}\cdot\vec{\sigma}_R+\vec{E}_{oI}\cdot\vec{\sigma}_I)+i(\vec{E}_{oI}\cdot\vec{\sigma}_R-\vec{E}_{oR}\cdot\vec{\sigma}_I)\right](\vec{E}_{oR}-i\vec{E}_{oI})$. Therefore,

$$\mathrm{Re}\,[(\vec{E}_o\cdot\vec{\sigma}^*)\vec{E}_o^*] = (\vec{E}_{oR}\cdot\vec{\sigma}_R+\vec{E}_{oI}\cdot\vec{\sigma}_I)\vec{E}_{oR} + (\vec{E}_{oI}\cdot\vec{\sigma}_R-\vec{E}_{oR}\cdot\vec{\sigma}_I)\vec{E}_{oI}$$

From Maxwell's first equation, we know that $\vec{\sigma}\cdot\vec{E}_o = 0 \Rightarrow$

$(\vec{\sigma}_R+i\vec{\sigma}_I)\cdot(\vec{E}_{oR}+i\vec{E}_{oI}) = 0 \Rightarrow \vec{\sigma}_R\cdot\vec{E}_{oR}-\vec{\sigma}_I\cdot\vec{E}_{oI}=0$ and $\vec{\sigma}_R\cdot\vec{E}_{oI}+\vec{\sigma}_I\cdot\vec{E}_{oR}=0$.

Therefore, $\vec{\sigma}_R\cdot\vec{E}_{oR} = \vec{\sigma}_I\cdot\vec{E}_{oI}$ and $\vec{\sigma}_R\cdot\vec{E}_{oI} = -\vec{\sigma}_I\cdot\vec{E}_{oR}$. We thus have:

$$\mathrm{Re}\left[(\vec{E}_o\cdot\vec{\sigma}^*)\vec{E}_o^*\right] = 2(\vec{E}_{oI}\cdot\vec{\sigma}_I)\vec{E}_{oR} - 2(\vec{E}_{oR}\cdot\vec{\sigma}_I)\vec{E}_{oI} = 2\vec{\sigma}_I\times(\vec{E}_{oR}\times\vec{E}_{oI})$$

The time-averaged Poynting vector may thus be written as follows:

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\exp(-2k_o\vec{\sigma}_I\cdot\vec{r})}{2Z_o}\left\{(|\vec{E}_{oR}|^2+|\vec{E}_{oI}|^2)\vec{\sigma}_R + 2(\vec{E}_{oR}\times\vec{E}_{oI})\times\vec{\sigma}_I\right\}$$

The last term of the above expression will be equal to zero under the following conditions:

1) $\vec{\sigma}_I = 0$, i.e., the plane-wave is homogeneous. A necessary (but not sufficient) condition for this to happen is $\epsilon_I(\omega) = 0$, since $\vec{\sigma}\cdot\vec{\sigma} = \epsilon(\omega)$ $\Rightarrow 2\vec{\sigma}_R\cdot\vec{\sigma}_I = \epsilon_I(\omega)$. Therefore, $\vec{\sigma}_I = 0$ automatically implies that $\epsilon_I(\omega) = 0$.

2) $\vec{E}_{oR} \times \vec{E}_{oI} = 0$, that is, $\vec{E}_{oR} \parallel \vec{E}_{oI}$ or either $\vec{E}_{oR} = 0$ or $\vec{E}_{oI} = 0$. In all such cases the plane-wave is linearly polarized. Thus elliptical polarization is necessary (although not sufficient) for the last term to be non-zero.

3) $(\vec{E}_{oR} \times \vec{E}_{oI}) \times \vec{\sigma}_I = 0$, that is, $\vec{\sigma}_I$ is $\perp$ to the plane of the ellipse of polarization. This implies that $\vec{E}_{oR}\cdot\vec{\sigma}_I = 0$ and $\vec{E}_{oI}\cdot\vec{\sigma}_I = 0$. But $\vec{\sigma}\cdot\vec{E}_o = 0 \Rightarrow \vec{\sigma}_R\cdot\vec{E}_{oR} = \vec{\sigma}_I\cdot\vec{E}_{oI}$ and $\vec{\sigma}_R\cdot\vec{E}_{oI} = -\vec{\sigma}_I\cdot\vec{E}_{oR}$. Therefore, $\vec{\sigma}_R$ must also be $\perp$ to the plane of the ellipse and, therefore, parallel to $\vec{\sigma}_I$. Now, when $\vec{\sigma}_R \parallel \vec{\sigma}_I$ it is always the case that they are $\perp$ to the ellipse of polarization (because $\vec{\sigma}\cdot\vec{E}_o = 0$). Thus, for the last term in the expression for $\langle \vec{S}(\vec{r},t)\rangle$ to be equal to zero, we find that $\vec{\sigma}_R \parallel \vec{\sigma}_I$ is both necessary and sufficient, so long as the beam is elliptically polarized. (According to (2) above, in the case of linear polarization, the last term is zero, irrespective of the relation between $\vec{\sigma}_R$ and $\vec{\sigma}_I$.). When $\vec{\sigma}_R \parallel \vec{\sigma}_I$ we'll have $\vec{\sigma}_R\cdot\vec{\sigma}_I = |\vec{\sigma}_R|\,|\vec{\sigma}_I|$. Therefore,

$$\vec{\sigma}\cdot\vec{\sigma} = \epsilon(\omega) \Rightarrow (\vec{\sigma}_R + i\vec{\sigma}_I)\cdot(\vec{\sigma}_R + i\vec{\sigma}_I) = (n_R + i n_I)^2 \Rightarrow$$

$$|\vec{\sigma}_R|^2 - |\vec{\sigma}_I|^2 + 2i|\vec{\sigma}_R||\vec{\sigma}_I| = n_R^2 - n_I^2 + 2i n_R n_I \Rightarrow |\vec{\sigma}_R| = n_R \text{ and } |\vec{\sigma}_I| = n_I.$$