Problem 22) The Fresnel reflection coefficients of Eqs.(17a) and (19a) should be used here in the special case when the incidence medium is glass of refractive index n, that is, $\mu_a(\omega) = 1.0$, $\varepsilon_a(\omega) = n^2$, while the transmittance medium is air, that is, $\mu_b(\omega) = \varepsilon_b(\omega) \cong 1.0$. Considering that total internal reflection occurs when $n \sin\theta > 1$, we write

$$\rho_p = \frac{n\sqrt{1-n^2\sin^2\theta} - \cos\theta}{n\sqrt{1-n^2\sin^2\theta} + \cos\theta} = -\frac{\cos\theta - \mathrm{i}n\sqrt{n^2\sin^2\theta-1}}{\cos\theta + \mathrm{i}n\sqrt{n^2\sin^2\theta-1}} = \exp(\mathrm{i}\pi)\exp\left[-\mathrm{i}2\tan^{-1}\left(\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}\right)\right],$$

$$\rho_s = \frac{n\cos\theta - \mathrm{i}\sqrt{n^2\sin^2\theta-1}}{n\cos\theta + \mathrm{i}\sqrt{n^2\sin^2\theta-1}} = \exp\left[-\mathrm{i}2\tan^{-1}\left(\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}\right)\right].$$

The phase difference $\Delta\varphi = \varphi_p - \varphi_s$ between ρ_p and ρ_s is thus given by

$$\Delta\varphi = \pi - 2\tan^{-1}\left(\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}\right) + 2\tan^{-1}\left(\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}\right).$$

Consequently,

$$\tan\left(\frac{\pi}{2} - \frac{\Delta\varphi}{2}\right) = \tan\left[\tan^{-1}\left(\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}\right) - \tan^{-1}\left(\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}\right)\right]$$

Trigonometric identity: $\tan(\varphi_1 - \varphi_2) = \dfrac{\tan\varphi_1 - \tan\varphi_2}{1 + \tan\varphi_1\tan\varphi_2}$

$$= \frac{\left(\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}\right) - \left(\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}\right)}{1+\left(\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}\right)\left(\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}\right)} = \frac{(n^2-1)\cos\theta\sqrt{n^2\sin^2\theta-1}}{n\cos^2\theta + n(n^2\sin^2\theta-1)} = \frac{\cos\theta\sqrt{n^2\sin^2\theta-1}}{n\sin^2\theta}.$$

Invoking the identity $\tan\left(\frac{\pi}{2} - \frac{\Delta\varphi}{2}\right) = \cot(\Delta\varphi/2) = 1/\tan(\Delta\varphi/2)$, we finally arrive at

$$\tan(\Delta\varphi/2) = \frac{n\sin^2\theta}{\cos\theta\sqrt{n^2\sin^2\theta-1}}.$$

Note: Occasionally, the inverse of the above formula appears in the literature, the reason being that the authors have defined ρ_p with a minus sign in front, thus eliminating π from the phase φ_p of ρ_p.

As θ rises beyond the critical TIR angle $\theta_c = \sin^{-1}(1/n)$ to grazing incidence at $\theta = 90°$, $\Delta\varphi$ drops from an initial value of $180°$ to a minimum, then rises again to $180°$. The minimum is found by setting the derivative with respect to θ of the above function equal to zero, that is,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\tan(\Delta\varphi/2) = 0 \quad \rightarrow \quad \cos^2\theta_{\min} = n^2\sin^2\theta_{\min} - 1 \quad \rightarrow \quad \tan^2\theta_{\min} = \frac{2}{n^2-1}.$$

The minimum $\Delta\varphi$ thus occurs at the incidence angle $\theta_{\min} = \tan^{-1}\sqrt{2/(n^2-1)}$, where $\tan(\Delta\varphi_{\min}/2) = 2n/(n^2-1)$.

For $n = 1.5$, we find $\theta_{\min} = 51.67°$ and $\Delta\varphi_{\min} = 134.76°$; therefore, a $\Delta\varphi$ of $45°$ is not feasible. However, $\Delta\varphi = 135°$ is just as good a choice for the Fresnel rhomb, and this may be achieved with two incidence angles on either side of $\theta_{\min}$, namely, $\theta_1 = 53.25°$ and $\theta_2 = 50.23°$. Either of these incidence angles will result, upon total internal reflection, in a phase-shift of $\Delta\varphi = \varphi_p - \varphi_s = 135°$. When two successive reflections occur at the same angle θ, the overall phase shift will be cumulative, i.e., $\Delta\varphi_{\text{total}} = 270°$, which is the same as $\Delta\varphi_{\text{total}} = -90°$. The Table below lists the overall phase shift for three different values of n at either θ_1 or θ_2.

	$\Delta\varphi_{\text{total}} = 2(\varphi_{\text{p}} - \varphi_{\text{s}})$		
θ_{inc}	$(n = 1.47)$	$(n = 1.5)$	$(n = 1.52)$
$53.25°$	$273.87°$	$270°$	$267.66°$
$50.23°$	$275.30°$	$270°$	$266.86°$

Clearly, for $\theta_1 = 53.25°$ the range of variation of $\Delta\varphi_{\text{total}}$ is smaller than that for $\theta_2 = 50.23°$. The better choice for the Fresnel rhomb, therefore, is $\theta_1 = 53.25°$.