Opti 501 **Solutions**

Problem 21) a) $\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = -\mathrm{Re}\left\{ i\psi_0 \vec{k} e^{i(\vec{k}\cdot\vec{r}-\omega t)} - i\omega \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$

$$\Rightarrow \vec{E}(\vec{r},t) = \mathrm{Im}\left\{ (\psi_0 \vec{k} - \omega \vec{A}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}.$$

b) $\vec{B}(\vec{r},t) = \mu_0 \mu(\omega) \vec{H}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = \mathrm{Re}\left\{ i\vec{k}\times\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} \Rightarrow$

$$\vec{H}(\vec{r},t) = -\frac{1}{\mu_0 \mu(\omega)} \mathrm{Im}\left\{ \vec{k}\times\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}.$$

c) 1) $\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) = 0 \Rightarrow \epsilon_0 \epsilon \vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0 \Rightarrow \mathrm{Im}\left\{ i\vec{k}\cdot(\psi_0\vec{k} - \omega\vec{A}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} = 0$

$$\Rightarrow \vec{k}\cdot(\psi_0 \vec{k} - \omega \vec{A}_0) = 0 \Rightarrow k^2 \psi_0 = \omega \vec{k}\cdot\vec{A}_0$$

2) $\vec{\nabla}\times\vec{H}(\vec{r},t) = \vec{J}_{free} + \frac{\partial \vec{D}(\vec{r},t)}{\partial t} \Rightarrow -\frac{1}{\mu_0 \mu(\omega)} \mathrm{Im}\left\{ i\vec{k}\times(\vec{k}\times\vec{A}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} =$

$\epsilon_0 \epsilon(\omega) \mathrm{Im}\left\{ -i\omega(\psi_0 \vec{k} - \omega\vec{A}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} \Rightarrow$

$\mathrm{Re}\left\{ \left[\vec{k}\times(\vec{k}\times\vec{A}_0) - \mu_0\epsilon_0 \mu(\omega)\epsilon(\omega)\,\omega(\psi_0\vec{k} - \omega\vec{A}_0) \right] e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} = 0 \Rightarrow$

$(\vec{k}\cdot\vec{A}_0)\vec{k} - k^2\vec{A}_0 - (\omega/c^2)\mu(\omega)\epsilon(\omega)\psi_0\vec{k} + (\omega^2/c^2)\mu(\omega)\epsilon(\omega)\vec{A}_0 = 0 \Rightarrow$

using the result obtained above

$(k^2\psi_0/\omega)\vec{k} - (\omega/c^2)\mu(\omega)\epsilon(\omega)\psi_0\vec{k} = [k^2 - (\omega/c)^2\mu(\omega)\epsilon(\omega)]\vec{A}_0 \Rightarrow$

$[k^2 - (\omega/c)^2\mu(\omega)\epsilon(\omega)](\psi_0/\omega)\vec{k} = [k^2 - (\omega/c)^2\mu(\omega)\epsilon(\omega)]\vec{A}_0$

The last equation is valid for the component of $\vec{A}_0$ parallel to $\vec{k}$. To see this, dot-multiply both sides with $\vec{k}$ and compare with the result obtained from Maxwell's 1st equation earlier. However, for any component of $\vec{A}_0$ that is not parallel to $\vec{k}$, the only way the equation can be satisfied is if $k^2 = (\omega/c)^2\mu(\omega)\epsilon(\omega)$.

3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \mathrm{Im}\left\{ i\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = -\mathrm{Re}\left\{(-i\omega)\, i(\vec{k}\times\vec{A}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$

$$\Rightarrow \mathrm{Re}\left\{\left[\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)+\omega(\vec{k}\times\vec{A}_0)\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \quad \Rightarrow$$

$$\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)+\omega\vec{k}\times\vec{A}_0 = 0 \Rightarrow \psi_0\cancelto{0}{(\vec{k}\times\vec{k})} - \omega\vec{k}\times\vec{A}_0 + \omega\vec{k}\times\vec{A}_0 = 0 \;\checkmark$$

Maxwell's 3rd Equation is thus automatically satisfied by the choice of $\psi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$.

4) $\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0 \Rightarrow \mathrm{Re}\left\{(i\vec{k})\cdot(i\vec{k}\times\vec{A}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow \vec{k}\cdot(\vec{k}\times\vec{A}_0) = 0$

$$\Rightarrow (\vec{k}\times\vec{k})\cdot\vec{A}_0 = 0 \quad \leftarrow \text{Automatically satisfied.}$$

Thus, to satisfy all four equations of Maxwell, we must have $k^2 = (\omega/c)^2\mu(\omega)\epsilon(\omega)$ and $k^2\psi_0 = \omega\vec{k}\cdot\vec{A}_0$. The latter requirement may also be written as $\vec{k}\cdot\vec{A}_0 = (\frac{\omega}{c^2})\mu(\omega)\epsilon(\omega)\psi_0$.

d) Lorenz gauge: $\vec{\nabla}\cdot\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\psi(\vec{r},t) = 0 \Rightarrow \mathrm{Re}\left\{ i\vec{k}\cdot\vec{A}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} +$

$$\frac{1}{c^2}\mathrm{Re}\left\{-i\omega\psi_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow \mathrm{Re}\left\{ i(\vec{k}\cdot\vec{A}_0 - \frac{\omega}{c^2}\psi_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0$$

$$\Rightarrow \vec{k}\cdot\vec{A}_0 - (\frac{\omega}{c^2})\psi_0 = 0 \Rightarrow \vec{k}\cdot\vec{A}_0 = (\frac{\omega}{c^2})\psi_0 \quad \leftarrow \text{Required for the Lorenz gauge.}$$

It is thus seen that the Lorenz gauge is not generally satisfied. Exceptions occur when $\mu(\omega)\epsilon(\omega) = 1$, or when $\psi_0 = 0$.