Opti 501 **Solutions**

Problem 20) Maxwell's equations in an isotropic, homogeneous, linear medium:

1) $\vec{\nabla}\cdot\vec{D}=0 \Rightarrow \vec{k}\cdot\vec{E}_0=0$

2) $\vec{\nabla}\times\vec{H}=\partial\vec{D}/\partial t \Rightarrow \vec{k}\times\vec{H}_0=-\omega\epsilon_0\epsilon\vec{E}_0$

3) $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t \Rightarrow \vec{k}\times\vec{E}_0=\omega\mu_0\mu\vec{H}_0$

$\Rightarrow \vec{k}\times(\vec{k}\times\vec{E}_0)=-\omega^2\mu_0\epsilon_0\mu\epsilon\vec{E}_0 \Rightarrow$

$\underbrace{(\vec{k}\cdot\vec{E}_0)}_{=0 \,\leftarrow\, \text{from Eq.(1)}}\vec{E}_0-k^2\vec{E}_0=-(\omega/c)^2\mu\epsilon\vec{E}_0$

4) $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{k}\cdot\vec{H}_0=0$

$\Rightarrow k^2=\vec{k}\cdot\vec{k}=(\omega/c)^2\mu(\omega)\epsilon(\omega)$.

a) Since $\vec{k}$ is real-valued, and $\mu(\omega)$ and $\epsilon(\omega)$ are both real and positive, we have $k^2=(\vec{k}'+i\vec{k}'')\cdot(\vec{k}'+i\vec{k}'')=k'^2-k''^2+2i\vec{k}'\cdot\vec{k}''=k'^2=(\omega/c)^2\mu(\omega)\epsilon(\omega)$

$\Rightarrow k=|\vec{k}|=(\omega/c)\sqrt{\mu(\omega)\epsilon(\omega)}$.

b) From Maxwell's first equation $\vec{k}\cdot\vec{E}_0=0$. Since $\vec{k}$ is real-valued we have $\vec{k}\cdot(\vec{E}_0'+i\vec{E}_0'')=0 \Rightarrow \vec{k}\cdot\vec{E}_0'=0$ and $\vec{k}\cdot\vec{E}_0''=0$. Therefore, $\vec{k}\perp\vec{E}_0'$ and $\vec{k}\perp\vec{E}_0''$

c) From Maxwell's fourth equation $\vec{k}\cdot\vec{H}_0=0$. Since $\vec{k}$ is real-valued we have: $\vec{k}\cdot(\vec{H}_0'+i\vec{H}_0'')=0 \Rightarrow \vec{k}\cdot\vec{H}_0'=0$ and $\vec{k}\cdot\vec{H}_0''=0$. Therefore, $\vec{k}\perp\vec{H}_0'$ and $\vec{k}\perp\vec{H}_0''$.

d) From Maxwell's third equation $\vec{k}\times\vec{E}_0=\omega\mu_0\mu\vec{H}_0 \Rightarrow$ Since $\vec{k}$ is real-valued and also because $\mu(\omega)$ is real-valued

$\vec{k}\times\vec{E}_0'+i\vec{k}\times\vec{E}_0''=\omega\mu_0\mu(\vec{H}_0'+i\vec{H}_0'') \Rightarrow$

$$\left\{\begin{array}{l} \vec{H}_0'=\dfrac{\vec{k}\times\vec{E}_0'}{\omega\mu_0\mu} \\ \vec{H}_0''=\dfrac{\vec{k}\times\vec{E}_0''}{\omega\mu_0\mu}\end{array}\right. \xrightarrow[\text{and also } \vec{k}\perp\vec{E}_0'']{\text{Since } \vec{k}\perp\vec{E}_0'} \left\{\begin{array}{l} \vec{H}_0'\perp\vec{E}_0' \text{ and } H_0'=\dfrac{(\omega/c)\sqrt{\mu\epsilon}}{\omega\mu_0\mu}E_0'=\dfrac{E_0'}{Z_0\sqrt{\mu/\epsilon}} \\ \vec{H}_0''\perp\vec{E}_0'' \text{ and } H_0''=\dfrac{E_0''}{Z_0\sqrt{\mu/\epsilon}}\end{array}\right.$$

e) $\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=[\vec{E}_0'\cos(\vec{k}\cdot\vec{r}-\omega t)-\vec{E}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]\times[\vec{H}_0'\cos(\vec{k}\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]$

$=(\vec{E}_0'\times\vec{H}_0')\cos^2(\vec{k}\cdot\vec{r}-\omega t)+(\vec{E}_0''\times\vec{H}_0'')\sin^2(\vec{k}\cdot\vec{r}-\omega t)-\tfrac{1}{2}(\vec{E}_0'\times\vec{H}_0''+\vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$

$$= \frac{E_0'^2}{Z_0\sqrt{\mu/\epsilon}}\,\hat{k}\cos^2(\vec{k}\cdot\vec{r}-\omega t) + \frac{E_0''^2\,\hat{k}}{Z_0\sqrt{\mu/\epsilon}}\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \frac{\vec{E}_0'\cdot\vec{E}_0'' + \vec{E}_0''\cdot\vec{E}_0'}{2Z_0\sqrt{\mu/\epsilon}}\,\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)].$$

Defining the real and positive refractive index $n(\omega) = \sqrt{\mu(\omega)\epsilon(\omega)}$, we can rewrite the above expression as follows:

$$\vec{S}(\vec{r},t) = \frac{\hat{k}}{2Z_0\sqrt{\mu/\epsilon}}\left\{(E_0'^2+E_0''^2) + (E_0'^2-E_0''^2)\cos\left[2\omega\left(t-\frac{n\hat{k}\cdot\vec{r}}{c}\right)\right] + 2\vec{E}_0'\cdot\vec{E}_0''\sin\left[2\omega\left(t-\frac{n\hat{k}\cdot\vec{r}}{c}\right)\right]\right\}.$$

The energy propagates along the direction of the unit-vector $\hat{k} = \vec{k}/k$.