Opti 501 **Solutions**

Problem 19) Maxwell's equations in free space:

1) $\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \vec{k}\cdot\vec{E}_o=0$
2) $\vec{\nabla}\times\vec{H}=\epsilon_o\partial\vec{E}/\partial t \Rightarrow \vec{k}\times\vec{H}_o=-\epsilon_o\omega\vec{E}_o$
3) $\vec{\nabla}\times\vec{E}=-\mu_o\partial\vec{H}/\partial t \Rightarrow \vec{k}\times\vec{E}_o=+\mu_o\omega\vec{H}_o$
4) $\vec{\nabla}\cdot\vec{H}=0 \Rightarrow \vec{k}\cdot\vec{H}_o=0$

$$\Rightarrow \vec{k}\times(\vec{k}\times\vec{E}_o)=-\mu_o\epsilon_o\omega^2\vec{E}_o \Rightarrow \underbrace{(\vec{k}\cdot\vec{E}_o)}_{=0 \text{ from Eq.(1)}}\vec{k}-k^2\vec{E}_o=-(\omega/c)^2\vec{E}_o \Rightarrow k^2=(\omega/c)^2$$

a) $\vec{k}=k_x\hat{x}+ik_z\hat{z} \Rightarrow k^2=\vec{k}\cdot\vec{k}=(k_x\hat{x}+ik_z\hat{z})\cdot(k_x\hat{x}+ik_z\hat{z})=k_x^2-k_z^2=(\omega/c)^2.$

b) From Maxwell's 1st equation: $\vec{k}\cdot\vec{E}_o=0 \Rightarrow (k_x\hat{x}+ik_z\hat{z})\cdot(E_{xo}\hat{x}+iE_{zo}\hat{z})=0 \Rightarrow$
$k_xE_{xo}-k_zE_{zo}=0 \Rightarrow k_xE_{xo}=k_zE_{zo}.$

c) From Maxwell's 3rd equation: $\vec{k}\times\vec{E}_o=\mu_o\omega\vec{H}_o \Rightarrow (k_x\hat{x}+ik_z\hat{z})\times(E_{xo}\hat{x}+iE_{zo}\hat{z})$
$=i\mu_o\omega H_{yo}\hat{y} \Rightarrow ik_xE_{zo}(\hat{x}\times\hat{z})+ik_zE_{xo}(\hat{z}\times\hat{x})=i(k_zE_{xo}-k_xE_{zo})\hat{y}=i\mu_o\omega H_{yo}\hat{y}$

$$\Rightarrow H_{yo}=\frac{k_zE_{xo}-k_xE_{zo}}{\mu_o\omega}=\frac{k_z-(k_x^2/k_z)}{\mu_o\omega}E_{xo}=\frac{k_z^2-k_x^2}{\mu_o\omega k_z}E_{xo}=-\frac{(\omega/c)^2}{\mu_o\omega\sqrt{k_x^2-\omega^2/c^2}}E_{xo}$$

$$\Rightarrow H_{yo}=-\frac{\omega/c}{\mu_o\omega\sqrt{(ck_x/\omega)^2-1}}E_{xo} \Rightarrow H_{yo}=-\frac{E_{xo}}{Z_o\sqrt{(\frac{ck_x}{\omega})^2-1}}$$

In the above derivation we chose the positive sign for $k_z=\sqrt{k_x^2-(\omega/c)^2}$, the reason being that, the field-amplitudes are proportional to $\exp(i\vec{k}\cdot\vec{r})=$ $\exp[i(k_x\hat{x}+ik_z\hat{z})\cdot\vec{r}]=\exp(-k_zz)\exp(ik_xx)$. When $z\to+\infty$ the fields can't become indefinitely large; therefore, $k_z\geq 0$.

d) $\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\,\mathrm{Re}\{\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\} = \frac{1}{2}\,\mathrm{Re}\{(E_{x_0}\hat{x}+iE_{z_0}\hat{z})e^{i(k_x\hat{x}+ik_z\hat{z})\cdot\vec{r}}\times$

$(-iH_{y_0}\hat{y})e^{-i(k_x\hat{x}-ik_z\hat{z})\cdot\vec{r}}\} = \frac{1}{2}e^{-2k_z z}\,\mathrm{Re}\{(E_{x_0}\hat{x}+iE_{z_0}\hat{z})\times(-iH_{y_0}\hat{y})\}$

$= \frac{1}{2}e^{-2k_z z}\,\mathrm{Re}\{-iE_{x_0}H_{y_0}\hat{z}-E_{z_0}H_{y_0}\hat{x}\} = \frac{-1}{2}\exp(-2k_z z)E_{z_0}H_{y_0}\hat{x}$

$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\exp(-2k_z z)\frac{E_{x_0}E_{z_0}\hat{x}}{Z_0\sqrt{(ck_x/\omega)^2-1}} = \frac{k_x E_{x_0}^2\exp(-2k_z z)}{2Z_0 k_z\sqrt{(ck_x/\omega)^2-1}}\,\hat{x} \Rightarrow$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{(ck_x/\omega)\exp[-2(\omega/c)\sqrt{(ck_x/\omega)^2-1}\,z]}{2Z_0[(ck_x/\omega)^2-1]}\,E_{x_0}^2\,\hat{x}.$$

The energy flows along the positive x-axis when $k_x > 0$.