Problem 18) a) $\vec{E}_0 = \vec{E}_0' + i\vec{E}_0''$. When $\vec{E}_0'$ and $\vec{E}_0''$ are <u>parallel</u> to each other, i.e., when they point in the same direction, the beam is <u>linearly polarized</u>. Let's define the unit vector $\hat{u}$ as the common direction of $\vec{E}_0'$ and $\vec{E}_0''$. We'll have:

$$\vec{E}(\vec{r},t) = \vec{E}_0'\cos(\vec{k}\cdot\vec{r}-\omega t) + \vec{E}_0''\sin(\vec{k}\cdot\vec{r}-\omega t) = \sqrt{E_0'^2+E_0''^2}\,[\cos\phi_0\cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$+\sin\phi_0\sin(\vec{k}\cdot\vec{r}-\omega t)]\,\hat{u} = \sqrt{E_0'^2+E_0''^2}\,\cos[\vec{k}\cdot\vec{r}-\omega t+\phi_0]\,\hat{u}$$

The last expression represents a linearly-polarized plane-wave, with E-field along the $\hat{u}$ direction. The phase-angle ϕ_0 is related to E_0' and E_0'' as follows: $\tan\phi_0 = E_0''/E_0'$.

b) $\vec{E}_0 = \vec{E}_0' + i\vec{E}_0''$. The plane-wave is circularly-polarized when $\vec{E}_0' \perp \vec{E}_0''$ <u>and</u> $|\vec{E}_0'| = |\vec{E}_0''|$. At a fixed point in space, say $\vec{r}=\vec{r}_0$, when $t=\vec{k}\cdot\vec{r}_0/\omega$ we have $\vec{E}(\vec{r}_0,t)=\vec{E}_0'$. A quarter of a period later (Note: Period $T=2\pi/\omega$), when $t=\frac{1}{4}T+(\vec{k}\cdot\vec{r}_0/\omega)$, we'll have $\vec{E}(\vec{r}_0,t)=\vec{E}_0''$. By definition, the E-field of a circularly-polarized wave rotates uniformly at frequency ω, covering a quarter of the circle during each quarter-period, $T/4$. Therefore, $\vec{E}_0'$ and $\vec{E}_0''$ must have equal length and be perpendicular to each other.

c) Maxwell's equations in free-space:

1) $\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \vec{k}\cdot\vec{E}_0=0$

2) $\vec{\nabla}\times\vec{H} = \epsilon_0\partial\vec{E}/\partial t \Rightarrow \vec{k}\times\vec{H}_0 = -\epsilon_0\omega\vec{E}_0$

3) $\vec{\nabla}\times\vec{E} = -\mu_0\partial\vec{H}/\partial t \Rightarrow \vec{k}\times\vec{E}_0 = \mu_0\omega\vec{H}_0$

4) $\vec{\nabla}\cdot\vec{H}=0 \Rightarrow \vec{k}\cdot\vec{H}_0=0$

From 2) and 3):

$$\vec{k}\times(\vec{k}\times\vec{E}_0) = -\mu_0\epsilon_0\omega^2\vec{E}_0 \Rightarrow \underbrace{\cancel{(\vec{k}\cdot\vec{E}_0)}}_{0}\vec{k} - (\vec{k}\cdot\vec{k})\vec{E}_0 = -(\omega/c)^2\vec{E}_0 \Rightarrow k^2=(\omega/c)^2$$

Maxwell's Eq.(1)

Since $\vec{k}$ for a homogeneous plane-wave is real valued, we have:

$$k^2 = \vec{k}\cdot\vec{k} = (\vec{k}'+i\vec{k}'')\cdot(\vec{k}'+i\vec{k}'') = k'^2 - k''^2 + 2i\,\vec{k}'\cdot\vec{k}'' = k'^2 \leftarrow \text{real}$$

Therefore, $k^2=(\omega/c)^2 \Rightarrow k=\omega/c.$

d) From Maxwell's equation 1, using the fact that $\vec{k} = \vec{k}' + i\vec{k}'' = \vec{k}'$, i.e., $\vec{k}$ is real-valued, we write:

$$\vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \vec{k}\cdot(\vec{E}_0' + i\vec{E}_0'') = 0 \Rightarrow \vec{k}\cdot\vec{E}_0' + i\vec{k}\cdot\vec{E}_0'' = 0 \Rightarrow \begin{cases}\vec{k}\cdot\vec{E}_0' = 0\\ \vec{k}\cdot\vec{E}_0'' = 0\end{cases}.$$

Therefore, $\vec{k} \perp \vec{E}_0'$ and $\vec{k} \perp \vec{E}_0''$.

e) From Maxwell's 4th equation $\vec{k}\cdot\vec{H}_0 = 0 \Rightarrow \vec{k}\cdot(\vec{H}_0' + i\vec{H}_0'') = 0 \Rightarrow \begin{cases}\vec{k}\cdot\vec{H}_0' = 0\\ \vec{k}\cdot\vec{H}_0'' = 0\end{cases}.$

Therefore, $\vec{k} \perp \vec{H}_0'$ and $\vec{k} \perp \vec{H}_0''$.

f) From Maxwell's 3rd equation $\vec{k}\times\vec{E}_0 = \mu_0\omega\vec{H}_0 \Rightarrow \vec{k}\times\vec{E}_0' + i\vec{k}\times\vec{E}_0'' = \mu_0\omega(\vec{H}_0' + i\vec{H}_0'')$

$$\Rightarrow \begin{cases}\vec{H}_0' = (\vec{k}\times\vec{E}_0')/\mu_0\omega\\ \vec{H}_0'' = (\vec{k}\times\vec{E}_0'')/\mu_0\omega\end{cases} \xrightarrow[\text{also } \vec{k}\perp\vec{E}_0'']{\text{Since } \vec{k}\perp\vec{E}_0' \text{ and}} \begin{cases}\vec{H}_0' \perp \vec{E}_0' \text{ and } H_0' = \frac{\omega/c}{\mu_0\omega}E_0' = E_0'/Z_0.\\ \vec{H}_0'' \perp \vec{E}_0'' \text{ and } H_0'' = \frac{\omega/c}{\mu_0\omega}E_0'' = E_0''/Z_0.\end{cases}$$

g) $\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) = [\vec{E}_0'\cos(\vec{k}\cdot\vec{r}-\omega t) - \vec{E}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]\times[\vec{H}_0'\cos(\vec{k}\cdot\vec{r}-\omega t) - \vec{H}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]$

$$= (\vec{E}_0'\times\vec{H}_0')\cos^2(\vec{k}\cdot\vec{r}-\omega t) + (\vec{E}_0''\times\vec{H}_0'')\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \tfrac{1}{2}(\vec{E}_0'\times\vec{H}_0'' + \vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$$

$$= \frac{E_0'^2}{Z_0}\hat{k}\cos^2(\vec{k}\cdot\vec{r}-\omega t) + \frac{E_0''^2}{Z_0}\hat{k}\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \frac{1}{2Z_0}(\vec{E}_0'\cdot\vec{E}_0'' + \vec{E}_0''\cdot\vec{E}_0')\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$$

$$\Rightarrow \vec{S}(\vec{r},t) = \frac{E_0'^2 + E_0''^2}{2Z_0}\hat{k} + \frac{E_0'^2 - E_0''^2}{2Z_0}\hat{k}\cos[2(\vec{k}\cdot\vec{r}-\omega t)] - \frac{\vec{E}_0'\cdot\vec{E}_0''}{Z_0}\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)] \Rightarrow$$

$$\vec{S}(\vec{r},t) = \frac{\hat{k}}{2Z_0}\left\{(E_0'^2 + E_0''^2) + (E_0'^2 - E_0''^2)\cos\left[2\omega\left(t - \frac{\hat{k}\cdot\vec{r}}{c}\right)\right] + 2\vec{E}_0'\cdot\vec{E}_0''\sin\left[2\omega\left(t - \frac{\hat{k}\cdot\vec{r}}{c}\right)\right]\right\}$$

The energy propagates along $\hat{k}$.