Opti 501 **Solutions**

Problem 17)

a) $k=(\omega/c)\sqrt{\mu(\omega)\epsilon(\omega)} \Rightarrow k_1=(\omega_1/c)\sqrt{\mu(\omega_1)\epsilon(\omega_1)}\,, \quad k_2=(\omega_2/c)\sqrt{\mu(\omega_2)\epsilon(\omega_2)}\,.$

b) $$\frac{\partial}{\partial t}\mathcal{E}_E(x,t)=\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\left[E_1\cos(k_1x-\omega_1t)+E_2\cos(k_2x-\omega_2t)\right]\frac{\partial}{\partial t}\left[\epsilon_0\epsilon_1E_1\cos(k_1x-\omega_1t)\right.$$

$$\left.+\,\epsilon_0\epsilon_2E_2\cos(k_2x-\omega_2t)\right]=\epsilon_0\epsilon_1\omega_1E_1^2\cos(k_1x-\omega_1t)\sin(k_1x-\omega_1t)+$$

$$\epsilon_0\epsilon_2\omega_2E_2^2\cos(k_2x-\omega_2t)\sin(k_2x-\omega_2t)+\epsilon_0\epsilon_1\omega_1E_1E_2\sin(k_1x-\omega_1t)\cos(k_2x-\omega_2t)$$

$$+\,\epsilon_0\epsilon_2\omega_2E_1E_2\cos(k_1x-\omega_1t)\sin(k_2x-\omega_2t)\Rightarrow$$

$$\frac{\partial}{\partial t}\mathcal{E}_E(x,t)=\tfrac{1}{2}\epsilon_0\epsilon_1\omega_1E_1^2\sin[2(k_1x-\omega_1t)]+\tfrac{1}{2}\epsilon_0\epsilon_2\omega_2E_2^2\sin[2(k_2x-\omega_2t)]$$

$$+\tfrac{1}{2}\epsilon_0\epsilon_1\omega_1E_1E_2\left\{\sin[(k_1+k_2)x-(\omega_1+\omega_2)t]+\sin[(k_1-k_2)x-(\omega_1-\omega_2)t]\right\}$$

$$+\tfrac{1}{2}\epsilon_0\epsilon_2\omega_2E_1E_2\left\{\sin[(k_1+k_2)x-(\omega_1+\omega_2)t]-\sin[(k_1-k_2)x-(\omega_1-\omega_2)t]\right\}\Rightarrow$$

$$\frac{\partial\mathcal{E}_E(x,t)}{\partial t}=\tfrac{1}{2}\epsilon_0\epsilon_1\omega_1E_1^2\sin[2(k_1x-\omega_1t)]+\tfrac{1}{2}\epsilon_0\epsilon_2\omega_2E_2^2\sin[2(k_2x-\omega_2t)]$$

$$+\tfrac{1}{2}\epsilon_0(\epsilon_1\omega_1+\epsilon_2\omega_2)E_1E_2\sin[(k_1+k_2)x-(\omega_1+\omega_2)t]+\tfrac{1}{2}\epsilon_0(\epsilon_1\omega_1-\epsilon_2\omega_2)E_1E_2\sin[(k_1-k_2)x-(\omega_1-\omega_2)t]$$

$$\Rightarrow\mathcal{E}_E(x,t)=C_0+\tfrac{1}{4}\epsilon_0\epsilon_1E_1^2\cos[2(k_1x-\omega_1t)]+\tfrac{1}{4}\epsilon_0\epsilon_2E_2^2\cos[2(k_2x-\omega_2t)]$$

$$+\tfrac{1}{2}\epsilon_0\frac{\epsilon_1\omega_1+\epsilon_2\omega_2}{\omega_1+\omega_2}E_1E_2\cos[(k_1+k_2)x-(\omega_1+\omega_2)t]$$

$$+\tfrac{1}{2}\epsilon_0\frac{\epsilon_1\omega_1-\epsilon_2\omega_2}{\omega_1-\omega_2}E_1E_2\cos[(k_1-k_2)x-(\omega_1-\omega_2)t].$$

c) $$\frac{1}{T}\int_{t-T}^{t} \mathcal{E}_E(x,t')\,dt' = C_0 - \frac{1}{8T\omega_1}\epsilon_0\epsilon_1 E_1^2 \cancelto{0}{\sin[2(k_1x-\omega_1 t)]\Big|_{t-T}^{t}} - \frac{1}{8T\omega_2}\epsilon_0\epsilon_2 E_2^2 \cancelto{0}{\sin[2(k_2x-\omega_2 t)]\Big|_{t-T}^{t}}$$

$$-\frac{1}{2T}\epsilon_0\frac{\epsilon_1\omega_1+\epsilon_2\omega_2}{(\omega_1+\omega_2)^2}E_1E_2\sin[(k_1+k_2)x-(\omega_1+\omega_2)t]\Big|_{t-T}^{t} - \frac{\epsilon_0}{2T}\frac{\epsilon_1\omega_1-\epsilon_2\omega_2}{(\omega_1-\omega_2)^2}E_1E_2\sin[(k_1-k_2)x-(\omega_1-\omega_2)t]\Big|_{t-T}^{t}$$

Noting that $\omega_1 = m\Delta\omega$, $\omega_2=(m+1)\Delta\omega$, and $\omega_1+\omega_2=(2m+1)\Delta\omega$, we simplify the above as follows:

$$\frac{1}{T}\int_{t-T}^{t}\mathcal{E}_E(x,t')\,dt' = C_0 - \frac{\epsilon_0}{T}\frac{\epsilon(\omega_c)}{2\omega_c}E_1E_2\sin\left[\frac{2\omega_c n(\omega_c)}{c}x - 2\omega_c t\right]$$

$$+\frac{\epsilon_0}{T}\frac{\frac{d}{d\omega}[\omega\epsilon(\omega)]_{\omega_c}}{\Delta\omega}\hat{E_1}E_2\sin\left[\frac{\omega_1 n(\omega_1)-\omega_2 n(\omega_2)}{c}x-(\omega_1-\omega_2)t\right] \Rightarrow$$

$$\frac{1}{T}\int_{t-T}^{t}\mathcal{E}_E(x,t')\,dt' = C_0 - \frac{\epsilon_0 E_1E_2}{\pi}\left\{\cancelto{\simeq 0}{\frac{\epsilon(\omega_c)}{2m+1}\sin\left[\frac{2\omega_c n(\omega_c)}{c}x-2\omega_c t\right]} + \frac{d}{d\omega}[\omega\epsilon(\omega)]_{\omega_c}\sin\left[\frac{\Delta\omega}{c}n_g\left(x-\frac{ct}{n_g}\right)\right]\right\}$$

d) The first term can be neglected because $2m+1 = 2\omega_c/\Delta\omega$ is very large. The remaining term shows how the energy moves with the system. The velocity of energy is c/n_g, the group velocity.