Opti 501 **Solutions**

Problem 15) a) Snell's law: $k_x^{(\text{i})} = k_x^{(\text{t})}$. Below, both $k_x^{(\text{i})}$ and $k_x^{(\text{t})}$ will be written as k_x.

Dispersion relation in free space: $\boldsymbol{k}^{(\text{i})2} = k_x^{(\text{i})2} + k_z^{(\text{i})2} = (\omega/c)^2$; therefore, $k_z^{(\text{i})} = \pm\sqrt{(\omega/c)^2 - k_x^2}$. Note that, in general, the square root will yield a complex number. Either the plus sign or the minus sign (but not both) should be used for the square root.

Dispersion relation in material medium: $\boldsymbol{k}^{(\text{t})2} = k_x^{(\text{t})2} + k_z^{(\text{t})2} = (\omega/c)^2 \mu(\omega)\varepsilon(\omega)$. Since $k_x^{(\text{i})} = k_x^{(\text{t})} = k_x$ and $\mu(\omega) = 1$, we will have $k_z^{(\text{t})} = \pm\sqrt{(\omega/c)^2\varepsilon(\omega) - k_x^2}$. As before, the square root will, in general, yield a complex number. Either the plus sign or the minus sign (but not both) should be used.

b) Maxwell's first equation: $\boldsymbol{k}^{(\text{i})} \cdot \boldsymbol{E}_\text{o}^{(\text{i})} = 0 \rightarrow k_x^{(\text{i})} E_{xo}^{(\text{i})} + k_z^{(\text{i})} E_{zo}^{(\text{i})} = 0 \rightarrow E_{zo}^{(\text{i})} = -k_x E_{xo}^{(\text{i})}/k_z^{(\text{i})}$.

transmitted beam: $\boldsymbol{k}^{(\text{t})} \cdot \boldsymbol{E}_\text{o}^{(\text{t})} = 0 \rightarrow E_{zo}^{(\text{t})} = -k_x E_{xo}^{(\text{t})}/k_z^{(\text{t})}$.

Maxwell's third equation; incident beam: $\boldsymbol{k}^{(\text{i})} \times \boldsymbol{E}_\text{o}^{(\text{i})} = \mu_\text{o}\omega \boldsymbol{H}_\text{o}^{(\text{i})} \rightarrow H_{xo}^{(\text{i})} = -k_z^{(\text{i})} E_{yo}^{(\text{i})}/(\mu_\text{o}\omega)$;

$$H_{yo}^{(\text{i})} = [k_z^{(\text{i})} E_{xo}^{(\text{i})} - k_x E_{zo}^{(\text{i})}]/(\mu_\text{o}\omega) = \varepsilon_\text{o}\omega E_{xo}^{(\text{i})}/k_z^{(\text{i})}; \qquad H_{zo}^{(\text{i})} = k_x E_{yo}^{(\text{i})}/(\mu_\text{o}\omega).$$

transmitted beam: $\boldsymbol{k}^{(\text{t})} \times \boldsymbol{E}_\text{o}^{(\text{t})} = \mu_\text{o}\mu(\omega)\omega \boldsymbol{H}_\text{o}^{(\text{t})} \rightarrow H_{xo}^{(\text{t})} = -k_z^{(\text{t})} E_{yo}^{(\text{t})}/(\mu_\text{o}\omega)$;

$$H_{yo}^{(\text{t})} = [k_z^{(\text{t})} E_{xo}^{(\text{t})} - k_x E_{zo}^{(\text{t})}]/(\mu_\text{o}\omega) = \varepsilon_\text{o}\varepsilon\omega E_{xo}^{(\text{t})}/k_z^{(\text{t})}; \qquad H_{zo}^{(\text{t})} = k_x E_{yo}^{(\text{t})}/(\mu_\text{o}\omega).$$

c) Continuity equations for the tangential E- and H-fields at the $z=0$ interface:

$$p\text{-polarization:} \quad \begin{cases} E_{xo}^{(\text{i})} = E_{xo}^{(\text{t})} \\ H_{yo}^{(\text{i})} = H_{yo}^{(\text{t})} \rightarrow \ \varepsilon_\text{o}\omega E_{xo}^{(\text{i})}/k_z^{(\text{i})} = \varepsilon_\text{o}\varepsilon\omega E_{xo}^{(\text{t})}/k_z^{(\text{t})} \ \rightarrow \ k_z^{(\text{t})} = \varepsilon(\omega) k_z^{(\text{i})}. \end{cases}$$

$$s\text{-polarization:} \quad \begin{cases} E_{yo}^{(\text{i})} = E_{yo}^{(\text{t})} \\ H_{xo}^{(\text{i})} = H_{xo}^{(\text{t})} \ \rightarrow \ k_z^{(\text{t})} = k_z^{(\text{i})}. \end{cases}$$

d) For the case of p-polarization, satisfying the boundary conditions *without* a reflected wave requires that $k_z^{(\text{t})} = \varepsilon(\omega) k_z^{(\text{i})}$. Substituting in this equation the expressions for $k_z^{(\text{i})}$ and $k_z^{(\text{t})}$ obtained in part (a), we find

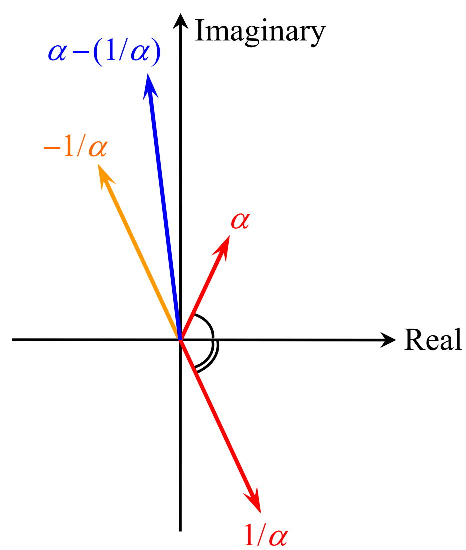
$$(\omega/c)^2\varepsilon(\omega) - k_x^2 = \varepsilon^2(\omega)[(\omega/c)^2 - k_x^2] \ \rightarrow \ k_x = \pm(\omega/c)\sqrt{\varepsilon(\omega)/[1+\varepsilon(\omega)]}.$$

For the case of s-polarization, the boundary conditions in the absence of a reflected wave will be satisfied only when $k_z^{(\text{i})} = k_z^{(\text{t})}$, which is *impossible* so long as $\varepsilon(\omega) \neq 1$.

e) **Case i**: $\varepsilon' > 0$, $\varepsilon'' = 0$. Here $\varepsilon' = n^2$, where n is the real-valued, positive refractive index of the material medium. When the reflection coefficient for p-polarized light vanishes, we will have $k_x = \pm(\omega/c)\sqrt{n^2/(1+n^2)} = \pm(\omega/c)\sin\theta_\text{B}$ where $\theta_\text{B} = \tan^{-1} n$ is the Brewster angle. Substituting for k_x in the expressions for $k_z^{(\text{i})}$ and $k_z^{(\text{t})}$, we find $k_z^{(\text{i})} = -(\omega/c)\cos\theta_\text{B}$ and $k_z^{(\text{t})} = -(n^2\omega/c)\cos\theta_\text{B}$. Both the incident and transmitted plane-waves are thus homogeneous; they propagate downward, along the negative z-axis, and satisfy the condition $k_z^{(\text{t})} = \varepsilon(\omega) k_z^{(\text{i})}$ obtained in part (c) for p-polarized light.

Case ii: $\varepsilon'<-1$, $\varepsilon''=0$. When the reflection coefficient for p-polarized light vanishes, we will have $k_x=\pm(\omega/c)\sqrt{|\varepsilon'|/(|\varepsilon'|-1)}$, which is a real-valued number with a magnitude greater than ω/c. Substitution for k_x in the expressions for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ yields $k_z^{(\mathrm{i})}=\mathrm{i}(\omega/c)/\sqrt{|\varepsilon'|-1}$ and $k_z^{(\mathrm{t})}=-\mathrm{i}(\omega/c)|\varepsilon'|/\sqrt{|\varepsilon'|-1}$. Both the incident and transmitted waves are thus *evanescent*, with real-valued k_x and imaginary k_z; they attenuate away from the interface along the $\pm z$-axis, and satisfy the required condition $k_z^{(\mathrm{t})}=\varepsilon(\omega)k_z^{(\mathrm{i})}$ obtained for p-polarized light in part (c). The time-averaged Poynting vector $\langle\boldsymbol{S}\rangle=\tfrac{1}{2}\mathrm{Real}(\boldsymbol{E}\times\boldsymbol{H}^*)$ can be readily calculated from the (E_x, E_z, H_y) fields given in part (b). The energy is seen to flow along k_x in the free space, and along $-k_x$ inside the medium. On both sides of the interface, the time-averaged energy flux along the z-axis is zero. This excited surface-wave, residing partly in the free space and partly in the material medium, is known as a *surface plasmon polariton*.

Case iii: $\varepsilon'<0$, $\varepsilon''>0$. In this case $k_x=\pm(\omega/c)\sqrt{(\varepsilon'+\mathrm{i}\varepsilon'')/(1+\varepsilon'+\mathrm{i}\varepsilon'')}$ is complex-valued. Substitution for k_x in the expressions for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ yields $k_z^{(\mathrm{i})}=(\omega/c)(1+\varepsilon'+\mathrm{i}\varepsilon'')^{-1/2}$ and $k_z^{(\mathrm{t})}=(\omega/c)(\varepsilon'+\mathrm{i}\varepsilon'')(1+\varepsilon'+\mathrm{i}\varepsilon'')^{-1/2}$. The complex square root $(1+\varepsilon'+\mathrm{i}\varepsilon'')^{-1/2}$ is chosen to give $k_z^{(\mathrm{i})}$ a *positive* imaginary part. Note that our choice of signs for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ satisfies the required condition $k_z^{(\mathrm{t})}=\varepsilon(\omega)k_z^{(\mathrm{i})}$ obtained in part (c). We must prove that the imaginary parts of $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ always have opposite signs. To this end, note that $(1+\varepsilon)^{-1/2}+\varepsilon(1+\varepsilon)^{-1/2}=(1+\varepsilon)^{1/2}$; therefore, $\varepsilon(1+\varepsilon)^{-1/2}=(1+\varepsilon)^{1/2}-(1+\varepsilon)^{-1/2}$. From the complex-plane diagram below it must be clear that, for *any* complex number α, the imaginary parts of $\alpha-(1/\alpha)$ and $(1/\alpha)$ always have opposite signs, which completes the proof. The *evanescent* plane-wave in the free space region decays exponentially along the imaginary part of $k_x^{(\mathrm{i})}\hat{\boldsymbol{x}}+k_z^{(\mathrm{i})}\hat{\boldsymbol{z}}$, which points away from the interface. The *inhomogeneous* plane-wave in the material medium also decays exponentially away from the interface, this one along the imaginary part of $k_x^{(\mathrm{t})}\hat{\boldsymbol{x}}+k_z^{(\mathrm{t})}\hat{\boldsymbol{z}}$.



Typical metals at optical frequencies have large negative values of ε' in addition to small positive values of ε''. For these, the *surface plasmon polariton* wave will have a k_x value slightly greater than unity (in magnitude), with a small imaginary component. The evanescent wave in the free space decays rather slowly along the z-axis, whereas the inhomogeneous wave in the metal decays quite rapidly away from the interface. The plasmonic wave is thus confined to a thin layer at the surface of the metallic medium. The time-averaged Poynting vector $\langle\boldsymbol{S}\rangle=\tfrac{1}{2}\mathrm{Real}(\boldsymbol{E}\times\boldsymbol{H}^*)$ can be readily calculated from the (E_x, E_z, H_y) fields given in part (b). The horizontal energy flux, $\langle S_x\rangle$, is seen to be along $\mathrm{Real}(k_x)$ in the free space, and along $\mathrm{Real}(-k_x)$ inside the medium. On both sides of the interface, vertical energy flux, $\langle S_z\rangle$, is downward, i.e., points along the negative z-axis. Such plasmonic waves are generally long-range, because ε'' is fairly small and the losses are confined to an exceedingly thin layer at the surface of the metallic medium.

Case iv: $\varepsilon' > 0$, $\varepsilon'' > 0$. This case is similar to case (iii), with the following exceptions: The magnitude of k_x is generally *less* than unity, with an imaginary part that may be large or small, depending on the relative values of ε' and ε''. For a low-loss medium, where ε'' is fairly small, the exponential decay of the wave inside the medium (away from the interface) is rather slow, resulting in a large penetration depth. The horizontal energy flux, $<S_x>$, is in the direction of $\text{Real}(k_x)$, both in the free space region and inside the material medium. The vertical energy flux, $<S_z>$, always pointing along the negative z-axis, is large, irrespective of whether ε'' is large or small. The wave is thus very different from a *surface plasmon polariton*, despite similarities in their mathematical structure. When integrated over the penetration depth, the lost energy will be substantial, even for small values of ε''. Therefore, a *p*-polarized wave-packet comprising an evanescent plane-wave in the free space region and an inhomogeneous plane-wave in a medium having $\varepsilon' > 0$, $\varepsilon'' > 0$, cannot behave similarly to a long-range surface wave; too much energy is dissipated within its penetration depth, and not enough energy is transported parallel to the surface of the medium.
