Opti 501 **Solutions**

Problem 14) a) Using the dispersion relation, $\boldsymbol{k}^2 = k_x^2 + k_z^2 = (\omega/c)^2\mu(\omega)\varepsilon(\omega)$, and the fact that the x-component of $\boldsymbol{k}$ is given by $k_x = (\omega/c)n(\omega)\sin\theta$, we write

$$k_z = \sqrt{(\omega/c)^2 n(\omega)^2 - k_x^2} = (\omega/c)n(\omega)\cos\theta. \qquad (1)$$

Maxwell's 1st equation: $\boldsymbol{k}_1\cdot\boldsymbol{E}_1 = 0 \rightarrow k_x E_{x1} + k_z E_{z1} = 0 \rightarrow E_{z1} = -k_x E_{x1}/k_z \rightarrow E_{z1} = -(\tan\theta)E_{x1}$.

$$\text{Similarly, } \boldsymbol{k}_2\cdot\boldsymbol{E}_2 = 0 \rightarrow E_{z2} = (\tan\theta)E_{x2}. \qquad (2)$$

Maxwell's 3rd equation: $\boldsymbol{k}_1\times\boldsymbol{E}_1 = \mu_o\mu(\omega)\omega\boldsymbol{H}_1 \rightarrow H_{x1} = -k_z E_{y1}/(\mu_o\omega) \rightarrow H_{x1} = -n(\omega)E_{y1}\cos\theta/Z_o;$

$$H_{y1} = (k_z E_{x1} - k_x E_{z1})/(\mu_o\omega) = n(\omega)E_{x1}/(Z_o\cos\theta); \quad H_{z1} = k_x E_{y1}/(\mu_o\omega) = n(\omega)E_{y1}\sin\theta/Z_o. \qquad (3a)$$

Similarly, $H_{x2} = n(\omega)E_{y2}\cos\theta/Z_o; \quad H_{y2} = -n(\omega)E_{x2}/(Z_o\cos\theta); \quad H_{z2} = n(\omega)E_{y2}\sin\theta/Z_o. \qquad (3b)$

b) Setting $E_{x2} = E_{x1}$ and $E_{y2} = E_{y1}$ for an even mode, the superposition of the two plane-waves produces the following fields throughout the waveguide:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{E}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)] + \boldsymbol{E}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\ &= \text{Real}\{[\boldsymbol{E}_1\exp(\mathrm{i}k_z z) + \boldsymbol{E}_2\exp(-\mathrm{i}k_z z)]\exp[\mathrm{i}(k_x x-\omega t)]\}\\ &= \text{Real}\{\{E_{x1}[\exp(\mathrm{i}k_z z)+\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{x}} + E_{y1}[\exp(\mathrm{i}k_z z)+\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{y}}\\ &\qquad -\tan\theta E_{x1}[\exp(\mathrm{i}k_z z)-\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{z}}\}\exp[\mathrm{i}(k_x x-\omega t)]\}\\ &= 2E_{x1}\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{x}} + 2E_{y1}\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{y}} + 2\tan\theta E_{x1}\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{z}}.\end{aligned} \qquad (4a)$$

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{H}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)] + \boldsymbol{H}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\ &= 2Z_o^{-1}n(\omega)[E_{y1}\cos\theta\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{x}} - (E_{x1}/\cos\theta)\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{y}}\\ &\qquad + E_{y1}\sin\theta\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{z}}].\end{aligned} \qquad (4b)$$

c) At the surface of the conductor, there cannot be any tangential E- or perpendicular B-fields, which means that $E_x = E_y = H_z = 0$ at $z = \pm d/2$. This is possible only when $\cos(\pm\tfrac{1}{2}k_z d) = 0$, that is,

$$\tfrac{1}{2}(\omega d/c)n(\omega)\cos\theta = (m+\tfrac{1}{2})\pi \quad\rightarrow\quad \cos\theta_m = (m+\tfrac{1}{2})\lambda_o/[n(\omega)d], \qquad (5)$$

where the vacuum wavelength, $\lambda_o = 2\pi c/\omega$, has been used. The mode can exist when $\cos\theta < 1$. The smallest possible value of the integer m being zero, it is necessary to have $d > \tfrac{1}{2}\lambda_o/n(\omega)$ to ensure the existence of at least one even mode. The even mode is said to be "cut-off" when the slab thickness d happens to be below $\tfrac{1}{2}\lambda_o/n(\omega)$. For single mode operation, d must be in the following range:

$$\tfrac{1}{2}\lambda_o/n(\omega) < d < \tfrac{3}{2}\lambda_o/n(\omega). \qquad (6)$$

With regard to the polarization state of the guided mode, two possibilities exist:

i) *p*-polarized mode (also called transverse magnetic, TM, mode): $E_{x1} \neq 0$, $E_{y1} = 0$.

ii) *s*-polarized mode (also called transverse electric, TE, mode): $E_{x1} = 0$, $E_{y1} \neq 0$.

In the case of *even* modes currently under consideration, the preceding statements with regard to cut-off and single-mode operation apply to *both* TE and TM modes.

d) According to Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$, the surface charge density is equal to the perpendicular D-field, $\varepsilon_o\varepsilon E_\perp$, at the surface of a perfect conductor. We thus have:

m^{th} p-polarized even mode:

$$\sigma_s(x,z=d/2,t)=-\varepsilon_o\varepsilon E_z(x,z=d/2,t)=2(-1)^{m+1}\varepsilon_o n^2(\omega)\tan\theta_m E_{x1}\sin(k_x^{(m)}x-\omega t). \qquad (7a)$$

m^{th} s-polarized even mode:

$$\sigma_s(x,z=d/2,t)=0. \qquad (7b)$$

Also, according to Maxwell's 2nd equation, $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{D}/\partial t$, the surface current density of a perfect conductor is equal but perpendicular to the tangential magnetic field, $\boldsymbol{H}_{||}$. Therefore,

m^{th} p-polarized even mode:

$$\boldsymbol{J}_s(x,z=d/2,t)=H_y(x,z=d/2,t)\hat{\boldsymbol{x}}=2(-1)^{m+1}n(\omega)(E_{x1}/Z_o\cos\theta_m)\sin(k_x^{(m)}x-\omega t)\hat{\boldsymbol{x}}. \qquad (8a)$$

m^{th} s-polarized even mode:

$$\boldsymbol{J}_s(x,z=d/2,t)=-H_x(x,z=d/2,t)\hat{\boldsymbol{y}}=2(-1)^{m+1}n(\omega)(E_{y1}/Z_o)\cos\theta_m\sin(k_x^{(m)}x-\omega t)\hat{\boldsymbol{y}}. \qquad (8b)$$

It may be readily verified that the above distributions satisfy the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s+\partial\sigma_s/\partial t=0$.

e) Setting $E_{x2}=-E_{x1}$ and $E_{y2}=-E_{y1}$ for an odd mode, the superposition of the two plane-waves produces the following fields throughout the waveguide:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t)&=\text{Real}\{\boldsymbol{E}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)]+\boldsymbol{E}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\&=\text{Real}\{\{E_{x1}[\exp(\mathrm{i}k_z z)-\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{x}}+E_{y1}[\exp(\mathrm{i}k_z z)-\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{y}}\\&\qquad-\tan\theta E_{x1}[\exp(\mathrm{i}k_z z)+\exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{z}}\}\exp[\mathrm{i}(k_x x-\omega t)]\}\\&=-2[E_{x1}\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{x}}+E_{y1}\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{y}}+\tan\theta E_{x1}\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{z}}].\end{aligned} \qquad (9a)$$

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r},t)&=\text{Real}\{\boldsymbol{H}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)]+\boldsymbol{H}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\&=-2Z_o^{-1}n(\omega)[E_{y1}\cos\theta\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{x}}-(E_{x1}/\cos\theta)\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{y}}\\&\qquad+E_{y1}\sin\theta\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{z}}].\end{aligned} \qquad (9b)$$

At the conductors' surfaces, $z=\pm d/2$, where $E_x=E_y=H_z=0$, we must have $\sin(\pm\tfrac{1}{2}k_z d)=0$, i.e.,

$$\tfrac{1}{2}(\omega d/c)n(\omega)\cos\theta=m\pi \quad\rightarrow\quad \cos\theta_m=m\lambda_o/[n(\omega)d]. \qquad (10)$$

In this case the lowest-order mode, corresponding to $m=0$, obtains when $\theta_m=90^\circ$. However, we now have $k_x=(\omega/c)n(\omega)$ and $k_z=0$. Under these circumstances, in accordance with Eqs.(9a) and (9b), E_x, E_y, H_x, and H_z will identically vanish throughout the slab. The only surviving fields are E_z and H_y, which go to infinity unless one recognizes that, by allowing E_x to approach zero when $\theta\rightarrow 90^\circ$, E_z and H_y could attain finite values, namely,

$$\boldsymbol{E}(\boldsymbol{r},t)=E_z\cos(k_x^{(0)}x-\omega t)\hat{\boldsymbol{z}}; \qquad (m=0), \qquad (11a)$$

$$\boldsymbol{H}(\mathbf{r},t) = -n(\omega)(E_z/Z_o)\cos(k_x^{(0)}x - \omega t)\hat{\boldsymbol{y}}; \qquad (m=0). \tag{11b}$$

This p-polarized (TM) mode always exists, no matter how thin the slab may be. Taking note of the fact that $\cos\theta_m \le 1$ for any value of m, the condition for p-polarized *single-mode* operation in the $m=0$ guided mode is $d < \lambda_o/n(\omega)$.

For odd modes that are s-polarized (TE), the first possibility for propagation is $m=1$, in which case single-mode operation occurs when $\lambda_o/n(\omega) < d < 2\lambda_o/n(\omega)$. The cut-off for *odd* TE modes occurs below $d = \lambda_o/n(\omega)$.

At the surface of the upper conductor which is in contact with the dielectric slab, surface charge and current densities for odd modes are found to be:

m^{th} odd p-polarized mode ($m \ne 0$):

$$\sigma_s(x, z=d/2, t) = -\varepsilon_o\varepsilon E_z(x, z=d/2, t) = 2(-1)^m\varepsilon_o n^2(\omega)\tan\theta_m E_{x1}\cos(k_x^{(m)}x - \omega t), \tag{12a}$$

$$\boldsymbol{J}_s(x, z=d/2, t) = H_y(x, z=d/2, t)\hat{\boldsymbol{x}} = 2(-1)^m n(\omega)(E_{x1}/Z_o\cos\theta_m)\cos(k_x^{(m)}x - \omega t)\hat{\boldsymbol{x}}. \tag{12b}$$

m^{th} odd s-polarized mode ($m \ne 0$):

$$\sigma_s(x, z=d/2, t) = 0, \tag{13a}$$

$$\boldsymbol{J}_s(x, z=d/2, t) = -H_x(x, z=d/2, t)\hat{\boldsymbol{y}} = 2(-1)^m n(\omega)(E_{y1}/Z_o)\cos\theta_m\cos(k_x^{(m)}x - \omega t)\hat{\boldsymbol{y}}. \tag{13b}$$

Once again, it is easy to verify the satisfaction of the charge-current continuity equation for the above distributions.
