Problem 12) Denoting by $\boldsymbol{\sigma}$ the normalized k-vector $\boldsymbol{k}/k_0 = c\boldsymbol{k}/\omega_0 = \lambda_0\boldsymbol{k}/2\pi$, we write

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \exp[\mathrm{i}(2\pi/\lambda_0)(\boldsymbol{\sigma}\cdot\boldsymbol{r} - ct)], \qquad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0 \exp[\mathrm{i}(2\pi/\lambda_0)(\boldsymbol{\sigma}\cdot\boldsymbol{r} - ct)].$$

Maxwell's first and fourth equations then yield

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{\sigma}\cdot\boldsymbol{E}_0 = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = \mu_0\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0 \quad\rightarrow\quad \boldsymbol{\sigma}\cdot\boldsymbol{H}_0 = 0.$$

Invoking Maxwell's second and third equations, we now find

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \mathrm{i}(2\pi/\lambda_0)\boldsymbol{\sigma}\times\boldsymbol{E}_0 = -(-\mathrm{i}2\pi c/\lambda_0)\mu_0\boldsymbol{H}_0 \;\rightarrow\; \boldsymbol{\sigma}\times\boldsymbol{E}_0 = Z_0\boldsymbol{H}_0.$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \partial\boldsymbol{D}/\partial t \quad\rightarrow\quad \mathrm{i}(2\pi/\lambda_0)\boldsymbol{\sigma}\times\boldsymbol{H}_0 = (-\mathrm{i}2\pi c/\lambda_0)\varepsilon_0\varepsilon\boldsymbol{E}_0 \;\rightarrow\; \boldsymbol{\sigma}\times\boldsymbol{H}_0 = -(\varepsilon/Z_0)\boldsymbol{E}_0.$$

Combining the above equations, we will have

$$\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{E}_0) = -\varepsilon\boldsymbol{E}_0 \quad\rightarrow\quad \cancelto{0}{(\boldsymbol{\sigma}\cdot\boldsymbol{E}_0)}\boldsymbol{\sigma} - (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\boldsymbol{E}_0 = -\varepsilon\boldsymbol{E}_0 \quad\rightarrow\quad \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \varepsilon = n^2.$$

a) In free space: $n = 1 \quad\rightarrow\quad \boldsymbol{\sigma} = \hat{\boldsymbol{z}} \;\rightarrow\; Z_0\boldsymbol{H}_\mathrm{i} = \hat{\boldsymbol{z}}\times E_\mathrm{i}\hat{\boldsymbol{x}} \quad\rightarrow\quad Z_0\boldsymbol{H}_\mathrm{i} = E_\mathrm{i}\hat{\boldsymbol{y}}.$

b) Inside the slab: $\boldsymbol{\sigma} = n\hat{\boldsymbol{z}} \;\rightarrow\; Z_0\boldsymbol{H}_\mathrm{t} = \boldsymbol{\sigma}\times\boldsymbol{E}_\mathrm{t} = n\hat{\boldsymbol{z}}\times\boldsymbol{E}_\mathrm{t} \quad\rightarrow\quad Z_0\boldsymbol{H}_\mathrm{t} = nE_\mathrm{t}\hat{\boldsymbol{y}}.$

c) Rate of energy flow in free space: $\langle\boldsymbol{S}_\mathrm{i}\rangle = ½\boldsymbol{E}_\mathrm{i}\times\boldsymbol{H}_\mathrm{i} = (E_\mathrm{i}^2/2Z_0)\hat{\boldsymbol{z}}.$

Rate of energy flow in the transparent slab: $\langle\boldsymbol{S}_\mathrm{t}\rangle = ½\boldsymbol{E}_\mathrm{t}\times\boldsymbol{H}_\mathrm{t} = (nE_\mathrm{t}^2/2Z_0)\hat{\boldsymbol{z}}.$

$$\langle\boldsymbol{S}_\mathrm{i}\rangle = \langle\boldsymbol{S}_\mathrm{t}\rangle \quad\rightarrow\quad E_\mathrm{i}^2/2Z_0 = nE_\mathrm{t}^2/2Z_0 \quad\rightarrow\quad E_\mathrm{t} = E_\mathrm{i}/\sqrt{n} \quad\rightarrow\quad H_\mathrm{t} = nE_\mathrm{t}/Z_0 = \sqrt{n}E_\mathrm{i}/Z_0.$$

d) E-field energy-density inside dispersionless medium: $¼\varepsilon_0\varepsilon E_\mathrm{t}^2 = ¼\varepsilon_0 n^2(E_\mathrm{i}/\sqrt{n})^2 = ¼\varepsilon_0 nE_\mathrm{i}^2.$

H-field energy-density inside dispersionless medium: $¼\mu_0 H_\mathrm{t}^2 = ¼\mu_0(\sqrt{n}\,E_\mathrm{i}/Z_0)^2 = ¼\varepsilon_0 nE_\mathrm{i}^2.$

Thus, the E- and H-field energy-densities within the transparent medium of the slab are equal.

Let the pulse duration and cross-sectional area be T and A, respectively. In the free-space region, the length of the pulse is cT, its volume is cTA, and its energy-density is $¼\varepsilon_0 E_\mathrm{i}^2 + ¼\mu_0 H_\mathrm{i}^2 = ¼\varepsilon_0 E_\mathrm{i}^2 + ¼\mu_0(E_\mathrm{i}/Z_0)^2 = ½\varepsilon_0 E_\mathrm{i}^2$. Consequently, the total energy of the light pulse in the free-space region is $½\varepsilon_0 E_\mathrm{i}^2 cTA$.

Inside the glass medium of the dielectric slab, the length of the pulse is cT/n, its volume is cTA/n, and its energy-density is $¼\varepsilon_0 nE_\mathrm{i}^2 + ¼\varepsilon_0 nE_\mathrm{i}^2 = ½\varepsilon_0 nE_\mathrm{i}^2$. Therefore, the total energy of the pulse propagating within the slab is also $½\varepsilon_0 E_\mathrm{i}^2 cTA$. The pulse energy is thus seen to be preserved.