Problem 11) a) The *E*- and *H*-fields of the incident plane-wave are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{po}^{(\mathrm{i})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)]; \quad (1a)$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{po}^{(\mathrm{i})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)]. \quad (1b)$$

The dispersion relation in free space is $k^2=(\omega/c)^2$. Therefore,

$$\boldsymbol{k}^{(\mathrm{i})}=k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}). \quad (2)$$

The incident *E*-field amplitude, as shown in the figure, is given by

$$\boldsymbol{E}_{po}^{(\mathrm{i})}=E_{po}(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}). \quad (3)$$

It may be readily verified that this *E*-field satisfies Maxwell's first equation, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, which is equivalent to $\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{E}_{po}^{(\mathrm{i})}=0$. As for the incident *H*-field, Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields

$$\mathrm{i}\boldsymbol{k}^{(\mathrm{i})}\times\boldsymbol{E}_{po}^{(\mathrm{i})}=\mathrm{i}\,\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{i})} \;\rightarrow\; (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}})\times E_{po}(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})=\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{i})}$$

$$\rightarrow\; \boldsymbol{H}_{po}^{(\mathrm{i})}=(E_{po}/Z_o)\,\hat{\boldsymbol{y}}. \quad (4)$$

b) The *E*- and *H*-fields of the reflected wave are written

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{po}^{(\mathrm{r})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r}-\omega t)]; \quad (5a)$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{po}^{(\mathrm{r})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r}-\omega t)]. \quad (5b)$$

The reflected *k*-vector is similar to the incident *k*-vector, except for the sign of k_z, that is,

$$\boldsymbol{k}^{(\mathrm{r})}=k_x\hat{\boldsymbol{x}}-k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}). \quad (6)$$

The reflected *E*-field amplitude must cancel out the tangential component of the incident *E*-field at the surface of the perfect conductor, as there cannot be any *E*-fields inside the conductor. We thus have

$$\boldsymbol{E}_{po}^{(\mathrm{r})}=-E_{po}(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}}). \quad (7)$$

As before, it may be readily verified that the above *E*-field satisfies Maxwell's first equation, namely, $\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{E}_{po}^{(\mathrm{r})}=0$. The reflected *H*-field is, once again, obtained from Maxwell's third equation, as follows:

$$\mathrm{i}\boldsymbol{k}^{(\mathrm{r})}\times\boldsymbol{E}_{po}^{(\mathrm{r})}=\mathrm{i}\,\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{r})} \;\rightarrow\; (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}})\times E_{po}(-\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})=\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{r})}$$

$$\rightarrow\; \boldsymbol{H}_{po}^{(\mathrm{r})}=(E_{po}/Z_o)\,\hat{\boldsymbol{y}}. \quad (8)$$

c) The rate of flow of energy per unit cross-sectional area per unit time is given by the time-averaged Poynting vector, namely,

$$< \boldsymbol{S}^{(\mathrm{i})}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}_{p\mathrm{o}}^{(\mathrm{i})}\times\boldsymbol{H}_{p\mathrm{o}}^{(\mathrm{i})*}] = \tfrac{1}{2}\mathrm{Re}[E_{p\mathrm{o}}(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}})\times(E_{p\mathrm{o}}^{*}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}]$$

$$= \frac{|E_{p\mathrm{o}}|^2}{2Z_{\mathrm{o}}}(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}}). \tag{9}$$

$$< \boldsymbol{S}^{(\mathrm{r})}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}_{p\mathrm{o}}^{(\mathrm{r})}\times\boldsymbol{H}_{p\mathrm{o}}^{(\mathrm{r})*}] = \tfrac{1}{2}\mathrm{Re}[-E_{p\mathrm{o}}(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}})\times(E_{p\mathrm{o}}^{*}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}]$$

$$= \frac{|E_{p\mathrm{o}}|^2}{2Z_{\mathrm{o}}}(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}}). \tag{10}$$

The incident and reflected waves are seen to have a time-averaged Poynting vector $<\boldsymbol{S}>$ directed along the corresponding k-vector. The magnitudes of these Poynting vectors, however, are the same, namely, $|E_{p\mathrm{o}}|^2/(2Z_{\mathrm{o}})$. Therefore, the incident and reflected energy fluxes are identical.

d) The surface-current-density $\boldsymbol{J}_s(x,y,t)$ is equal in magnitude and perpendicular in direction to the total H-field at the surface of the perfect conductor. Taking into account the right-hand rule relating the direction of the surface current to that of the H-field, we will have

$$\boldsymbol{J}_s(x,y,t) = [H_y^{(\mathrm{i})}(x,y,z=0,t) + H_y^{(\mathrm{r})}(x,y,z=0,t)]\,\hat{\boldsymbol{x}}$$

$$= 2(E_{p\mathrm{o}}/Z_{\mathrm{o}})\exp[\mathrm{i}(k_x x - \omega t)]\,\hat{\boldsymbol{x}} = 2(E_{p\mathrm{o}}/Z_{\mathrm{o}})\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]\,\hat{\boldsymbol{x}}. \tag{11}$$

e) The surface-charge-density $\sigma_s(x,y,t)$ is given by the discontinuity in the perpendicular component of the D-field, that is,

$$\sigma_s(x,y,t) = -\varepsilon_{\mathrm{o}}[E_z^{(\mathrm{i})}(x,y,z=0,t) + E_z^{(\mathrm{r})}(x,y,z=0,t)]$$

$$= 2\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(k_x x - \omega t)] = 2\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]. \tag{12}$$

f) Substituting in the continuity equation for $\boldsymbol{J}_s$ from Eq.(11) and for σ_s from Eq.(12), we find

$$\partial J_{sx}/\partial x + \partial\sigma_s/\partial t = [2\mathrm{i}(\omega/c)\sin\theta(E_{p\mathrm{o}}/Z_{\mathrm{o}}) - 2\mathrm{i}\,\omega\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta]\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]$$

$$= 2\mathrm{i}\omega[(cZ_{\mathrm{o}})^{-1} - \varepsilon_{\mathrm{o}}]E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)] = 0. \tag{13}$$

The continuity equation is thus satisfied by the induced surface-charge and surface-current.