Problem 10) a) The E- and H-fields of the incident plane-wave are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{so}^{(i)}\exp[\mathrm{i}(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r}-\omega t)]; \quad (1a)$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{so}^{(i)}\exp[\mathrm{i}(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r}-\omega t)]. \quad (1b)$$

The dispersion relation in free space is $k^2=(\omega/c)^2$. Therefore,

$$\boldsymbol{k}^{(i)}=k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}). \quad (2)$$

The incident E-field amplitude, as shown in the figure, is given by

$$\boldsymbol{E}_{so}^{(i)}=E_{so}^{(i)}\hat{\boldsymbol{y}}. \quad (3)$$

It may be readily verified that this E-field satisfies Maxwell's first equation, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, which is equivalent to $\boldsymbol{k}^{(i)}\cdot\boldsymbol{E}_{so}^{(i)}=0$. As for the incident H-field, Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields

$$\mathrm{i}\boldsymbol{k}^{(i)}\times\boldsymbol{E}_{so}^{(i)}=\mathrm{i}\,\omega\mu_o\boldsymbol{H}_{so}^{(i)} \;\rightarrow\; (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}})\times E_{so}^{(i)}\hat{\boldsymbol{y}}=\omega\mu_o\boldsymbol{H}_{so}^{(i)}$$

$$\rightarrow\; \boldsymbol{H}_{so}^{(i)}=-(E_{so}^{(i)}/Z_o)(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}). \quad (4)$$

A similar treatment yields for the reflected plane-wave,

$$\boldsymbol{k}^{(r)}=k_x\hat{\boldsymbol{x}}-k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}), \quad (5)$$

$$\boldsymbol{E}_{so}^{(r)}=E_{so}^{(r)}\hat{\boldsymbol{y}}, \quad (6)$$

$$\boldsymbol{H}_{so}^{(r)}=(E_{so}^{(r)}/Z_o)(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}}). \quad (7)$$

As for the transmitted beam, the dispersion relation in the dielectric medium is $k^2=(\omega/c)^2n^2(\omega)$; also, in accordance with Snell's law, we must have $k_x^{(t)}=k_x^{(i)}$ and $k_y^{(t)}=k_y^{(i)}=0$. Therefore,

$$\boldsymbol{k}^{(t)}=k_x^{(i)}\hat{\boldsymbol{x}}+k_z^{(t)}\hat{\boldsymbol{z}}=(\omega/c)[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}]. \quad (8)$$

Next, we obtain the transmitted E-field using the continuity of tangential $\boldsymbol{E}$ at the interface:

$$\boldsymbol{E}_{so}^{(t)}=(E_{so}^{(i)}+E_{so}^{(r)})\hat{\boldsymbol{y}}. \quad (9)$$

Subsequently, the transmitted H-field is obtained from Maxwell's third equation, as follows:

$$\boldsymbol{k}^{(t)}\times\boldsymbol{E}_{so}^{(t)}=\omega\mu_o\boldsymbol{H}_{so}^{(t)} \;\rightarrow\; (\omega/c)[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}]\times E_{so}^{(t)}\hat{\boldsymbol{y}}=\omega\mu_o\boldsymbol{H}_{so}^{(t)}$$

$$\rightarrow\; \boldsymbol{H}_{so}^{(t)}=-(E_{so}^{(t)}/Z_o)[\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}]. \quad (10)$$

b) Continuity of the tangential E-field is already assured by means of Eq. (9). The only remaining constraint involves the tangential H-field, whose continuity equation is written

$$H_x^{(i)}+H_x^{(r)}=H_x^{(t)} \;\rightarrow\; -(E_{so}^{(i)}/Z_o)\cos\theta+(E_{so}^{(r)}/Z_o)\cos\theta=-(E_{so}^{(t)}/Z_o)\sqrt{n^2(\omega)-\sin^2\theta}\,. \quad (11)$$

The Fresnel reflection and transmission coefficients, defined as $\rho_s=E_{so}^{(r)}/E_{so}^{(i)}$ and $\tau_s=E_{so}^{(t)}/E_{so}^{(i)}$, may now be used in conjunction with Eqs. (9) and (11) to yield

$$-E_{so}^{(i)}\cos\theta+\rho_s E_{so}^{(i)}\cos\theta=-(1+\rho_s)E_{so}^{(i)}\sqrt{n^2(\omega)-\sin^2\theta}\,. \tag{12}$$

Solving the above equation for ρ_s, we find

$$\rho_s=\frac{\cos\theta-\sqrt{n^2(\omega)-\sin^2\theta}}{\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}}. \tag{13}$$

From Eq.(9) and the definitions of the Fresnel coefficients, it is obvious that $\tau_s=1+\rho_s$; therefore,

$$\tau_s=\frac{2\cos\theta}{\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}}. \tag{14}$$

c) The rate-of-flow of energy per unit cross-sectional area per unit time for each of the three plane-waves is given by the corresponding time-averaged Poynting vector, as follows:

$$\langle \boldsymbol{S}^{(i)}(\boldsymbol{r},t)\rangle=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}_{so}^{(i)}\times\boldsymbol{H}_{so}^{(i)*}]=-\tfrac{1}{2}\mathrm{Re}[E_{so}^{(i)}\hat{\boldsymbol{y}}\times(E_{so}^{(i)*}/Z_o)(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})$$
$$=\frac{|E_{so}^{(i)}|^2}{2Z_o}(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}). \tag{15}$$

$$\langle \boldsymbol{S}^{(r)}(\boldsymbol{r},t)\rangle=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}_{so}^{(r)}\times\boldsymbol{H}_{so}^{(r)*}]=\tfrac{1}{2}\mathrm{Re}[E_{so}^{(r)}\hat{\boldsymbol{y}}\times(E_{so}^{(r)*}/Z_o)(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}})$$
$$=\frac{|E_{so}^{(r)}|^2}{2Z_o}(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}})=|\rho_s|^2\frac{|E_{so}^{(i)}|^2}{2Z_o}(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}). \tag{16}$$

$$\langle \boldsymbol{S}^{(t)}(\boldsymbol{r},t)\rangle=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}_{so}^{(t)}\times\boldsymbol{H}_{so}^{(t)*}]=-\tfrac{1}{2}\mathrm{Re}[E_{so}^{(t)}\hat{\boldsymbol{y}}\times(E_{so}^{(t)*}/Z_o)[\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}]$$
$$=\frac{|E_{so}^{(t)}|^2}{2Z_o}[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}]$$
$$=|\tau_s|^2\frac{n(\omega)|E_{so}^{(i)}|^2}{2Z_o}(\sin\theta'\,\hat{\boldsymbol{x}}+\cos\theta'\,\hat{\boldsymbol{z}}). \tag{17}$$

To verify the conservation of energy, consider an incident beam whose cross-sectional diameter in the xz-plane is D. The footprint of this beam on the x-axis will then be $D/\cos\theta$, resulting in a transmitted beam whose cross-sectional diameter in the xz-plane is $D(\cos\theta'/\cos\theta)$. Considering the various Poynting vectors in Eqs.(15)-(17), and the fact that the reflected beam diameter in the xz-plane remains equal to D, we must show that the following identity holds:

$$|\rho_s|^2+(\cos\theta'/\cos\theta)\,n(\omega)\,|\tau_s|^2=1. \tag{18}$$

Substitution from Eqs.(13) and (14) into Eq.(18), and noting that $n(\omega)\cos\theta'=\sqrt{n^2(\omega)-\sin^2\theta}$, then yields

$$\frac{[\cos\theta-\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}{[\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}+\frac{[\sqrt{n^2(\omega)-\sin^2\theta}\,/\cos\theta]\,(2\cos\theta)^2}{[\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}=1. \tag{19}$$

The energy fluxes of the reflected and transmitted beams thus add up to that of he incident beam, proving that electromagnetic energy in the present problem is conserved.