Problem 7.9)

a) At normal incidence, where $\theta = 0$, we find

$$\rho_p = \frac{n_1\sqrt{\varepsilon_2} - \varepsilon_2}{n_1\sqrt{\varepsilon_2} + \varepsilon_2} = \frac{n_1 - \sqrt{\varepsilon_2}}{n_1 + \sqrt{\varepsilon_2}}, \qquad \rho_s = \frac{n_1 - \sqrt{\varepsilon_2}}{n_1 + \sqrt{\varepsilon_2}}.$$

Thus $\rho_p = \rho_s$ at normal incidence. This is expected, of course, because at normal incidence there is no difference between p- and s-polarization. In other words, at $\theta = 0$ the plane of incidence, relative to which p- and s-directions are defined, becomes indeterminate.

b) TIR occurs when the second medium is also transparent, that is, when ε_2 is real-valued and positive. In TIR we must have $|\rho_p| = 1$ and $|\rho_s| = 1$. Each Fresnel reflection coefficient is the ratio of the difference to the sum of two numbers. When two complex numbers have a phase difference of 90°, their sum and difference will have exactly the same magnitudes and, therefore, the ratio of difference to sum will have a magnitude of 1.0. In the formulas for ρ_p and ρ_s, since n_1, ε_2, $\sin\theta$ and $\cos\theta$ are all real numbers, the only way in which the numbers in the numerator and denominator could acquire a 90° phase difference would be for the square-roots to become purely imaginary. Therefore, when $\varepsilon_2 - n_1^2 \sin^2\theta < 0$, the square-roots become imaginary, the two numbers appearing in the numerator and denominator of each expression for the Fresnel coefficient become 90° apart, and we obtain $|\rho_p| = |\rho_s| = 1$. The condition for TIR is thus

$$\varepsilon_2 - n_1^2 \sin^2\theta < 0 \quad \rightarrow \quad \sin^2\theta > \varepsilon_2/n_1^2 = n_2^2/n_1^2 \quad \rightarrow \quad \sin\theta > n_2/n_1.$$

Considering that $\sin\theta$ cannot exceed unity, we must have $n_2 < n_1$. The critical angle for TIR is readily seen to be $\theta_c = \sin^{-1}(n_2/n_1)$.

From the above argument it must be clear that the critical angles for p- and s-polarized plane-waves are the *same*. One can also argue on physical grounds that the two critical angles must coincide, as follows. For $|\rho_p| = 1$ and/or $|\rho_s| = 1$, we must have 100% of the incident optical energy reflected back into the incidence medium. This means that the transmitted beam in both cases (i.e., for p- and s-polarized light) cannot carry any energy and must, therefore, be *evanescent*. Now, the condition for evanescence is a condition on the k-vector of the transmitted plane-wave, *not* on its E- and H-field amplitudes. When the boundary conditions are matched, the k-vector is determined by the requirement that k_x and k_y (and also ω) be the same for the incident, reflected, and transmitted plane-waves. This requirement (which is rooted in Maxwell's boundary conditions) is what we have referred to as Snell's law. Thus, transition from propagating to evanescent field occurs as a result of the application of Snell's law to the k-vectors, which is independent of the incident beam being p- or s-polarized. The condition for TIR, therefore, cannot depend on the state of polarization of the incident beam.

c) At Brewster's angle $\rho_p = 0$, that is,

$$n_1\sqrt{\varepsilon_2 - n_1^2 \sin^2\theta} = \varepsilon_2 \cos\theta \quad \rightarrow \quad n_1^2(\varepsilon_2 - n_1^2 \sin^2\theta) = \varepsilon_2^2 \cos^2\theta$$

$$\rightarrow \quad \frac{n_1^2 \varepsilon_2}{\varepsilon_2^2 \cos^2\theta} - \frac{n_1^4 \sin^2\theta}{\varepsilon_2^2 \cos^2\theta} = 1 \quad \rightarrow \quad (n_1^2/\varepsilon_2)(1 + \tan^2\theta) - (n_1^2/\varepsilon_2)^2 \tan^2\theta = 1$$

$$\rightarrow \quad \cancel{(n_1^2/\varepsilon_2)} - 1 = (n_1^2/\varepsilon_2)[\cancel{(n_1^2/\varepsilon_2)} - 1]\tan^2\theta \qquad \rightarrow \qquad \tan^2\theta = \varepsilon_2/n_1^2$$

$$\rightarrow \quad \tan\theta = (n_2 + \mathrm{i}\kappa_2)/n_1.$$

In the above derivation, we have ignored the trivial solution, $n_1^2/\varepsilon_2 = 1$, which yields $n_1 = \sqrt{\varepsilon_2}$, and results in a perfect "impedance match" at the interface.

For a non-trivial solution to exist, we must have $\tan\theta = (n_2/n_1) + \mathrm{i}(\kappa_2/n_1)$. Given that $\tan\theta$ is real, κ_2 must be zero. Moreover, $0 \le \theta < 90°$ requires that $\tan\theta > 0$; consequently, $n_2 > 0$. Thus, a Brewster's angle exists when ε_2 is real and positive. The incidence angle at which $\rho_p = 0$ is seen to be $\theta_\mathrm{B} = \tan^{-1}(\sqrt{\varepsilon_2}/n_1)$. Both $n_2 = \sqrt{\varepsilon_2} > n_1$ and $n_2 < n_1$ are allowed.

For s-polarized light, a Brewster's angle does not exist because

$$\rho_s = 0 \quad \rightarrow \quad n_1\cos\theta = \sqrt{\varepsilon_2 - n_1^2\sin^2\theta} \quad \rightarrow \quad n_1^2(\cos^2\theta + \sin^2\theta) = \varepsilon_2 \quad \rightarrow \quad n_1^2 = \varepsilon_2,$$

which is true only in the trivial case when there is perfect "impedance match" at the interface between the two media.

d) We have already seen in part (c) above that for ρ_p to become zero, κ_2 must vanish. Thus any absorption (or gain for that matter) is incompatible with the existence of a Brewster's angle.

e) When $\theta \rightarrow 90°$, $\sin\theta \rightarrow 1$ and $\cos\theta \rightarrow 0$. Therefore,

$$\rho_p = \left(n_1\sqrt{\varepsilon_2 - n_1^2}\right)/\left(n_1\sqrt{\varepsilon_2 - n_1^2}\right) = +1,$$

$$\rho_s = -\sqrt{\varepsilon_2 - n_1^2}/\sqrt{\varepsilon_2 - n_1^2} = -1.$$

With s-light, the incident and reflected plane-waves cancel each other out, and there will be no E-field parallel to the surface, nor will there be an H-field perpendicular to the surface. The same thing happens with p-light; however, one must be careful in interpreting the meaning of ρ_p, since it represents the ratio $E_{x0}^\mathrm{r}/E_{x0}^\mathrm{i}$. Indeed, the ratio $E_{z0}^\mathrm{r}/E_{z0}^\mathrm{i}$ approaches -1 for p-light, as does the ratio $H_{y0}^\mathrm{r}/H_{y0}^\mathrm{i}$.

f) The answer is *yes*, there can be total reflection at the interface when $\kappa_2 \neq 0$, but then we must have $n_2 = 0$; in other words, ε_2 must be real-valued and negative. The proof is outlined below.

i) Case of s-polarization. In order to have $|\rho_s|^2 = 1.0$, we must have

$$\left(n_1\cos\theta - \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\right)\left(n_1\cos\theta - \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\right)^*$$

$$= \left(n_1\cos\theta + \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\right)\left(n_1\cos\theta + \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\right)^*$$

$$\rightarrow -n_1\cos\theta\left(\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^*\right) = n_1\cos\theta\left(\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^*\right)$$

$$\rightarrow \sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^* = 0 \qquad \rightarrow \qquad \sqrt{\varepsilon_2 - n_1^2\sin^2\theta} = \text{purely imaginary}.$$

Let α be an arbitrary real number. Then $\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} = \mathrm{i}\alpha$ yields $\varepsilon_2 = n_1^2\sin^2\theta - \alpha^2$, which is a purely real number. If ε_2 happens to be a *positive* real number, we will have the condition for conventional TIR at the interface between two transparent media, namely, $\sin\theta > \sqrt{\varepsilon_2}/n_1 = n_2/n_1$. However, if ε_2 turns out to be a *negative* real number, then $\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}$ will be imaginary for all values of θ, resulting in 100% reflectivity at *all* incidence angles θ.

ii) Case of $\boldsymbol{p}$-polarization. In order to have $|\rho_p|^2 = 1.0$, we must have

$$\left(n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} - \varepsilon_2\cos\theta\right)\left(n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} - \varepsilon_2\cos\theta\right)^* = \left(n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \varepsilon_2\cos\theta\right)\left(n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \varepsilon_2\cos\theta\right)^*$$

$$\rightarrow \quad -n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\,\varepsilon_2^*\cos\theta - \varepsilon_2\cos\theta\, n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^{\,*} = n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}\,\varepsilon_2^*\cos\theta + \varepsilon_2\cos\theta\, n_1\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^{\,*}$$

$$\rightarrow \quad \varepsilon_2^*\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} + \varepsilon_2\sqrt{\varepsilon_2 - n_1^2\sin^2\theta}^{\,*} = 0 \;\rightarrow\; \varepsilon_2^*\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} = \text{purely imaginary}.$$

Let α be an arbitrary real number. Then $\varepsilon_2^*\sqrt{\varepsilon_2 - n_1^2\sin^2\theta} = \mathrm{i}\alpha$ yields

$$(\varepsilon_2' - \mathrm{i}\varepsilon_2'')^2(\varepsilon_2' - n_1^2\sin^2\theta + \mathrm{i}\varepsilon_2'') = -\alpha^2$$

$$\rightarrow \quad (\varepsilon_2'^2 - \varepsilon_2''^2 - 2\mathrm{i}\varepsilon_2'\varepsilon_2'')(\varepsilon_2' - n_1^2\sin^2\theta + \mathrm{i}\varepsilon_2'') = -\alpha^2 \quad (\alpha \text{ is an arbitrary real number})$$

$$\rightarrow \quad \begin{cases}(\varepsilon_2'^2 - \varepsilon_2''^2)(\varepsilon_2' - n_1^2\sin^2\theta) + 2\varepsilon_2'\varepsilon_2''^2 = -\alpha^2\\ \varepsilon_2''(\varepsilon_2'^2 - \varepsilon_2''^2) - 2\varepsilon_2'\varepsilon_2''(\varepsilon_2' - n_1^2\sin^2\theta) = 0\end{cases}$$

$$\rightarrow \quad \begin{cases}\varepsilon_2'^2(\varepsilon_2' - n_1^2\sin^2\theta) + \varepsilon_2''^2(\varepsilon_2' + n_1^2\sin^2\theta) = -\alpha^2\\ \varepsilon_2''(\varepsilon_2''^2 + \varepsilon_2'^2 - 2\varepsilon_2' n_1^2\sin^2\theta) = 0 \;\rightarrow\; \begin{cases}\varepsilon_2'' = 0 & \text{(first solution)}\\ \varepsilon_2''^2 = 2\varepsilon_2' n_1^2\sin^2\theta - \varepsilon_2'^2 & \text{(second solution)}\end{cases}\end{cases}$$

First solution: $\varepsilon_2'' = 0 \;\rightarrow\; \varepsilon_2'^2(\varepsilon_2' - n_1^2\sin^2\theta) = -\alpha^2$

$$\rightarrow\; \varepsilon_2' - n_1^2\sin^2\theta = \text{negative real number} \;\rightarrow\; \begin{cases}\varepsilon_2' < 0,\\ 0 < \varepsilon_2' < n_1^2\sin^2\theta.\end{cases}$$

Therefore, two possibilities exist: (i) Conventional TIR, where ε_2 is real and positive, and $\sin^2\theta > \varepsilon_2/n_1^2 = (n_2/n_1)^2$, that is, $\sin\theta > n_2/n_1$. (ii) ε_2 is real and negative, in which case $|\rho_p| = 1$ for *all* angles of incidence θ.

Second solution: $\varepsilon_2''^2 = 2\varepsilon_2' n_1^2\sin^2\theta - \varepsilon_2'^2$

$$\rightarrow \quad \varepsilon_2'^3 - \varepsilon_2'^2 n_1^2\sin^2\theta + (2\varepsilon_2' n_1^2\sin^2\theta - \varepsilon_2'^2)(\varepsilon_2' + n_1^2\sin^2\theta) = -\alpha^2$$

$$\rightarrow \quad \cancel{\varepsilon_2'^3} - \cancel{\varepsilon_2'^2 n_1^2\sin^2\theta} + \cancel{2\varepsilon_2'^2 n_1^2\sin^2\theta} + 2\varepsilon_2' n_1^4\sin^4\theta - \cancel{\varepsilon_2'^3} - \cancel{\varepsilon_2'^2 n_1^2\sin^2\theta} = -\alpha^2$$

$$\rightarrow \quad 2\varepsilon_2' n_1^4\sin^4\theta = -\alpha^2 \quad \rightarrow \quad \varepsilon_2' \text{ is purely negative.}$$

However, if this happens we will have $\varepsilon_2''^2 = 2\varepsilon_2' n_1^2\sin^2\theta - \varepsilon_2'^2 < 0$, which is unacceptable for $\varepsilon_2''^2$, a purely positive number. Therefore, a second solution does not exist.
