Problem 7)

$\vec{k}\cdot\vec{k}=(\omega/c)^2\epsilon(\omega) \Rightarrow k_x^2+k_z^2=(\omega/c)^2\epsilon(\omega) \Rightarrow k_z=(\frac{\omega}{c})\sqrt{\epsilon(\omega)-(ck_x/\omega)^2}$; Both $\pm k_z$ are allowed.

$\vec{k}_a\cdot\vec{k}_a=(\omega/c)^2\epsilon_a(\omega) \Rightarrow k_x^2+k_{za}^2=(\omega/c)^2\epsilon_a(\omega) \Rightarrow k_{za}=(\omega/c)\sqrt{\epsilon_a(\omega)-(ck_x/\omega)^2}$; only $+k_{za}$ is allowed.

$\vec{k}_b\cdot\vec{k}_b=(\omega/c)^2\epsilon_b(\omega) \Rightarrow k_x^2+k_{zb}^2=(\omega/c)^2\epsilon_b(\omega) \Rightarrow k_{zb}=(\omega/c)\sqrt{\epsilon_b(\omega)-(ck_x/\omega)^2}$; only $-k_{zb}$ is allowed.

Waveguide's core: Dielectric Constant $=\epsilon(\omega)$; $\vec{k}_1=k_x\hat{x}+k_z\hat{z}$; $\vec{k}_2=k_x\hat{x}-k_z\hat{z}$.

$\vec{E}_1=E_{x1}\hat{x}+E_{y1}\hat{y}+E_{z1}\hat{z}$; $\vec{k}_1\cdot\vec{E}_1=0 \Rightarrow k_xE_{x1}+k_zE_{z1}=0 \Rightarrow E_{z1}=-(k_x/k_z)E_{x1}$.

$\vec{k}_1\times\vec{E}_1=\omega\mu_0\vec{H}_1 \Rightarrow (k_x\hat{x}+k_z\hat{z})\times(E_{x1}\hat{x}+E_{y1}\hat{y}-\frac{k_x}{k_z}E_{x1}\hat{z})=(\omega/c)Z_0\vec{H}_1 \Rightarrow$

$$(\omega/c)Z_0\vec{H}_1=-k_zE_{y1}\hat{x}+(k_zE_{x1}+\frac{k_x^2}{k_z}E_{x1})\hat{y}+k_xE_{y1}\hat{z}$$

$\vec{E}_2=E_{x2}\hat{x}+E_{y2}\hat{y}+E_{z2}\hat{z}$; $\vec{k}_2\cdot\vec{E}_2=0 \Rightarrow E_{z2}=+\frac{k_x}{k_z}E_{x2}$

$(\omega/c)Z_0\vec{H}_2=\vec{k}_2\times\vec{E}_2=k_zE_{y2}\hat{x}-(k_z+\frac{k_x^2}{k_z})E_{x2}\hat{y}+k_xE_{y2}\hat{z} \Rightarrow$

$$(\omega/c)Z_0\vec{H}_2=k_zE_{y2}\hat{x}-\frac{(\omega/c)^2\epsilon(\omega)}{k_z}E_{x2}\hat{y}+k_xE_{y2}\hat{z}.$$

Upper Cladding: Dielectric Constant $=\epsilon_a(\omega)$; $\vec{k}_a=k_x\hat{x}+k_{za}\hat{z}$

$\vec{E}_a=E_{xa}\hat{x}+E_{ya}\hat{y}+E_{za}\hat{z}$; $\vec{k}_a\cdot\vec{E}_a=0 \Rightarrow E_{za}=-\frac{k_x}{k_{za}}E_{xa}$

$(\omega/c)Z_0\vec{H}_a=\vec{k}_a\times\vec{E}_a=-k_{za}E_{ya}\hat{x}+\frac{(\omega/c)^2\epsilon_a(\omega)}{k_{za}}E_{xa}\hat{y}+k_xE_{ya}\hat{z}.$

Lower cladding: Dielectric Constant $= \epsilon_b(\omega)$; $\vec{k}_b = k_x\hat{x} - k_{zb}\hat{z}$

$\vec{E}_b = E_{xb}\hat{x} + E_{yb}\hat{y} + E_{zb}\hat{z}$; $\vec{k}_b \cdot \vec{E}_b = 0 \Rightarrow E_{zb} = (k_x/k_{zb})E_{xb}$

$$(\omega/c)Z_0\vec{H}_b = \vec{k}_b \times \vec{E}_b = k_{zb}E_{yb}\hat{x} - \frac{(\omega/c)^2\epsilon_b(\omega)}{k_{zb}}E_{xb}\hat{y} + k_xE_{yb}\hat{z}$$

The spatial and temporal dependence of all $\vec{E}$- and $\vec{H}$-fields is $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.

Boundary conditions at the upper interface:

① $E_{x1}e^{ik_zd/2} + E_{x2}e^{-ik_zd/2} = E_{xa}e^{ik_{za}d/2}$ ← Continuity of E_x

② $E_{y1}e^{ik_zd/2} + E_{y2}e^{-ik_zd/2} = E_{ya}e^{ik_{za}d/2}$ ← Continuity of E_y

③ $-k_zE_{y1}e^{ik_zd/2} + k_zE_{y2}e^{-ik_zd/2} = -k_{za}E_{ya}e^{ik_{za}d/2}$ ← Continuity of H_x

④ $\frac{\epsilon(\omega)}{k_z}E_{x1}e^{ik_zd/2} - \frac{\epsilon(\omega)}{k_z}E_{x2}e^{-ik_zd/2} = \frac{\epsilon_a(\omega)}{k_{za}}E_{xa}e^{ik_{za}d/2}$ ← Continuity of H_y

Boundary conditions at the lower interface:

⑤ $E_{x1}e^{-ik_zd/2} + E_{x2}e^{+ik_zd/2} = E_{xb}e^{+ik_{zb}d/2}$ ← Continuity of E_x

⑥ $E_{y1}e^{-ik_zd/2} + E_{y2}e^{+ik_zd/2} = E_{yb}e^{+ik_{zb}d/2}$ ← Continuity of E_y

⑦ $-k_zE_{y1}e^{-ik_zd/2} + k_zE_{y2}e^{+ik_zd/2} = +k_{zb}E_{yb}e^{+ik_{zb}d/2}$ ← Continuity of H_x

⑧ $\frac{\epsilon(\omega)}{k_z}E_{x1}e^{-ik_zd/2} - \frac{\epsilon(\omega)}{k_z}E_{x2}e^{+ik_zd/2} = -\frac{\epsilon_b(\omega)}{k_{zb}}E_{xb}e^{+ik_{zb}d/2}$ ← Continuity of H_y

From Eq. ① above we substitute for E_{xa} in Eq. ④. Similarly for E_{xb} from ⑤ in ⑧:

$$\begin{cases} \frac{\epsilon(\omega)}{k_z}E_{x1}e^{ik_zd/2} - \frac{\epsilon(\omega)}{k_z}E_{x2}e^{-ik_zd/2} = \frac{\epsilon_a(\omega)}{k_{za}}E_{x1}e^{ik_zd/2} + \frac{\epsilon_a(\omega)}{k_{za}}E_{x2}e^{-ik_zd/2} \\ \frac{\epsilon(\omega)}{k_z}E_{x1}e^{-ik_zd/2} - \frac{\epsilon(\omega)}{k_z}E_{x2}e^{+ik_zd/2} = -\frac{\epsilon_b(\omega)}{k_{zb}}E_{x1}e^{-ik_zd/2} - \frac{\epsilon_b(\omega)}{k_{zb}}E_{x2}e^{+ik_zd/2} \end{cases}$$

$$\Rightarrow \begin{cases} \left[\frac{\epsilon(\omega)}{k_z} - \frac{\epsilon_a(\omega)}{k_{za}}\right] E_{x1} e^{ik_z d} = \left[\frac{\epsilon(\omega)}{k_z} + \frac{\epsilon_a(\omega)}{k_{za}}\right] E_{x2} \\ \left[\frac{\epsilon(\omega)}{k_z} + \frac{\epsilon_b(\omega)}{k_{zb}}\right] E_{x1} = \left[\frac{\epsilon(\omega)}{k_z} - \frac{\epsilon_b(\omega)}{k_{zb}}\right] E_{x2} e^{ik_z d} \end{cases}$$

Next, we replace E_{x2} from the first into the second equation;

$$\left[\frac{\epsilon(\omega)}{k_z} + \frac{\epsilon_a(\omega)}{k_{za}}\right]\left[\frac{\epsilon(\omega)}{k_z} + \frac{\epsilon_b(\omega)}{k_{zb}}\right] \cancel{E_{x1}} = \left[\frac{\epsilon(\omega)}{k_z} - \frac{\epsilon_b(\omega)}{k_{zb}}\right]\left[\frac{\epsilon(\omega)}{k_z} - \frac{\epsilon_a(\omega)}{k_{za}}\right] \cancel{E_{x1}} e^{2ik_z d}$$

$$\Rightarrow \frac{\frac{\epsilon}{k_z} - \frac{\epsilon_a}{k_{za}}}{\frac{\epsilon}{k_z} + \frac{\epsilon_a}{k_{za}}} \cdot \frac{\frac{\epsilon}{k_z} - \frac{\epsilon_b}{k_{zb}}}{\frac{\epsilon}{k_z} + \frac{\epsilon_b}{k_{zb}}} e^{2ik_z d} = 1 \Rightarrow$$

$$\frac{\epsilon k_{za} - \epsilon_a k_z}{\epsilon k_{za} + \epsilon_a k_z} \cdot \frac{\epsilon k_{zb} - \epsilon_b k_z}{\epsilon k_{zb} + \epsilon_b k_z} \exp[2ik_z d] = 1$$ ← P-Polarized or TM light.

This is the <u>characteristic equation</u> for p-polarized (or TM) light. Since k_z, k_{za}, and k_{zb} can all be written in terms of k_x, this transcendental equation must be solved for k_x. Afterwards E_{x1} can be chosen arbitrarily, while E_{x2}, E_{xa}, and E_{xb} are computed from the preceding equations. Each solution k_x of the characteristic equation thus specifies a p-polarized (or TM) mode of the waveguide (aside from an arbitrary coefficient E_{x1} that defines the amplitude of the mode).

A similar procedure yields the characteristic equation for s-polarized (or TE) light. In this case equations 2 and 3 must be combined to eliminate E_{ya}. Then equations 6 and 7 are combined to eliminate E_{yb}. The remaining equations are solved for E_{y1} and E_{y2}. The characteristic equation for s-light is readily found as follows:

$$\frac{k_z - k_{za}}{k_z + k_{za}} \cdot \frac{k_z - k_{zb}}{k_z + k_{zb}} \exp(2ik_z d) = 1$$ ← s-polarized or TE light.

Again, this equation must be solved for k_x. Each solution will specify a particular TE mode of the waveguide. Aside from the arbitrary mode amplitude E_{y1}, all other field parameters will be determined once a value for k_x is specified.

When $\epsilon(\omega)$, $\epsilon_a(\omega)$, and $\epsilon_b(\omega)$ at the frequency of interest ω, are all real-valued and positive, the three media (i.e., core, upper cladding, lower cladding) will be transparent. In this case we set $k_x = (\omega/c)\sqrt{\epsilon(\omega)}\sin\theta = (\omega/c)\,n(\omega)\sin\theta$, where θ is the angle between the k-vector and the z-axis. We'll have:

$$\frac{\epsilon k_{za} - \epsilon_a k_z}{\epsilon k_{za} + \epsilon_a k_z} = \frac{n^2\sqrt{n_a^2 - n^2\sin^2\theta} - n_a^2\sqrt{n^2 - n^2\sin^2\theta}}{n^2\sqrt{n_a^2 - n^2\sin^2\theta} + n_a^2\sqrt{n^2 - n^2\sin^2\theta}} = \frac{n\sqrt{n_a^2 - n^2\sin^2\theta} - n_a^2\cos\theta}{n\sqrt{n_a^2 - n^2\sin^2\theta} + n_a^2\cos\theta}$$

$= \rho_p$ ← Fresnel reflection coefficient at the upper interface (p-light)

Also:

$$\frac{k_z - k_{za}}{k_z + k_{za}} = \frac{n\cos\theta - \sqrt{n_a^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n_a^2 - n^2\sin^2\theta}} = \rho_s$$ ← Fresnel reflection coefficient at the upper interface (s-light)

Similar relations may also be obtained for the lower interface. We then write the characteristic equations for the p- and s-polarized modes of the waveguide as follows (note that $\frac{\omega}{c}$ is written $\frac{2\pi}{\lambda_0}$):

$$\left\{\begin{array}{ll} \rho_p^{(top)}\rho_p^{(bottom)}\exp\left[i4\pi\left(\frac{nd}{\lambda_0}\right)\cos\theta\right] = 1 & \leftarrow \text{p-polarized (TM) modes} \\ \rho_s^{(top)}\rho_s^{(bottom)}\exp\left[i4\pi\left(\frac{nd}{\lambda_0}\right)\cos\theta\right] = 1 & \leftarrow \text{s-polarized (TE) modes} \end{array}\right.$$

The magnitudes of ρ_p and ρ_s <u>must</u> be equal to unity for the above equations to hold, that is, total internal reflection <u>must</u> occur at <u>both</u> interfaces. Thus for any modes at all to propagate within the slab waveguide it is necessary to have $k_x > (\omega/c)n_a$ and $k_x > (\omega/c)n_b$. In other words, guiding requires $n\sin\theta > \max(n_a, n_b)$.