Problem 5)

Incident beam: $\vec{k}^{(i)} = (\frac{\omega}{c})(\sin\theta\,\hat{x} - \cos\theta\,\hat{z})$ ✓

$$\vec{k}^{(i)}\cdot\vec{E}_o^{(i)} = 0 \Rightarrow (\sin\theta\,\hat{x} - \cos\theta\,\hat{z})\cdot(E_{xo}^{(i)}\hat{x} + E_{zo}^{(i)}\hat{z}) = 0 \Rightarrow$$

$$\sin\theta\, E_{xo}^{(i)} - \cos\theta\, E_{zo}^{(i)} = 0 \Rightarrow E_{zo}^{(i)} = \tan\theta\, E_{xo}^{(i)}$$ ✓

$$\vec{k}^{(i)}\times\vec{E}_o^{(i)} = \omega\mu_o\vec{H}_o^{(i)} \Rightarrow (\sin\theta\,\hat{x} - \cos\theta\,\hat{z})\times(E_{xo}^{(i)}\hat{x} + E_{zo}^{(i)}\hat{z}) = Z_o\vec{H}_o^{(i)} \Rightarrow$$

$$-(\sin\theta\, E_{zo}^{(i)} + \cos\theta\, E_{xo}^{(i)})\hat{y} = Z_o\vec{H}_o^{(i)} \Rightarrow Z_o\vec{H}_o^{(i)} = -(\sin\theta\tan\theta + \cos\theta)E_{xo}^{(i)}\hat{y}$$

$$\Rightarrow \vec{H}_o^{(i)} = -\frac{E_{xo}^{(i)}}{Z_o\cos\theta}\hat{y}.$$

Reflected beam: $\vec{k}^{(r)} = (\frac{\omega}{c})(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})$ ✓

$$\vec{k}^{(r)}\cdot\vec{E}_o^{(r)} = 0 \Rightarrow E_{zo}^{(r)} = -\tan\theta\, E_{xo}^{(r)}$$ ✓

$$\vec{k}^{(r)}\times\vec{E}_o^{(r)} = \omega\mu_o\vec{H}_o^{(r)} \Rightarrow \vec{H}_o^{(r)} = +\frac{rE_{xo}^{(i)}}{Z_o\cos\theta}\hat{y}.$$

The coefficients r, a, b, t used here differ from those in the statement of the problem. Here $r = E_x^{r}/E_x^{i}$, $a = E_x^{a}/E_x^{i}$, $b = E_x^{b}/E_x^{i}$ and $t = E_x^{t}/E_x^{i}$.

Beam $\underline{a}$ inside the dielectric coating layer: $\vec{k}^{(a)} = (\frac{\omega}{c})(\sin\theta\,\hat{x} - \sqrt{n_o^2 - \sin^2\theta}\,\hat{z})$ ✓

$$\vec{k}^{(a)}\cdot\vec{E}_o^{(a)} = 0 \Rightarrow \sin\theta\, E_{xo}^{(a)} - \sqrt{n_o^2 - \sin^2\theta}\, E_{zo}^{(a)} = 0 \Rightarrow E_{zo}^{(a)} = \frac{\sin\theta}{\sqrt{n_o^2 - \sin^2\theta}}E_{xo}^{(a)}$$ ✓

$$\vec{k}^{(a)}\times\vec{E}_o^{(a)} = \omega\mu_o\vec{H}_o^{(a)} \Rightarrow Z_o\vec{H}_o^{(a)} = -(\sin\theta\, E_{zo}^{(a)} + \sqrt{n_o^2 - \sin^2\theta}\, E_{xo}^{(a)})\hat{y} \Rightarrow$$

$$\vec{H}_o^{(a)} = -\frac{n_o^2\, a\, E_{xo}^{(i)}}{Z_o\sqrt{n_o^2 - \sin^2\theta}}\hat{y}$$ ✓

Beam $\underline{b}$ inside the dielectric coating layer: $\vec{k}^{(b)} = (\frac{\omega}{c})(\sin\theta\,\hat{x} + \sqrt{n_o^2 - \sin^2\theta}\,\hat{z})$ ✓

$$\vec{k}^{(b)}\cdot\vec{E}_o^{(b)} = 0 \Rightarrow E_{zo}^{(b)} = -\frac{\sin\theta}{\sqrt{n_o^2 - \sin^2\theta}}E_{xo}^{(b)}$$ ✓

$$\vec{k}^{(b)}\times\vec{E}_o^{(b)} = \omega\mu_o\vec{H}_o^{(b)} \Rightarrow \vec{H}_o^{(b)} = +\frac{n_o^2\, b\, E_{xo}^{(i)}}{Z_o\sqrt{n_o^2 - \sin^2\theta}}\hat{y}$$ ✓

Transmitted beam inside the substrate : $\vec{k}^{(t)} = (\frac{\omega}{c})(\sin\theta\,\hat{x} - \sqrt{n_s^2 - \sin^2\theta}\,\hat{z})$ ✓

$$\vec{k}^{(t)} \cdot \vec{E}_0^{(t)} = 0 \Rightarrow E_{z0}^{(t)} = \frac{\sin\theta}{\sqrt{n_s^2 - \sin^2\theta}} E_{x0}^{(t)} \quad ✓$$

$$\vec{k}^{(t)} \times \vec{E}_0^{(t)} = \omega\mu_0 \vec{H}_0^{(t)} \Rightarrow \vec{H}_0^{(t)} = -\frac{n_s^2 \tau E_{x0}^{(i)}}{Z_0\sqrt{n_s^2 - \sin^2\theta}}\,\hat{y} \quad ✓$$

Continuity of E_x and H_y at $z=0$:

E-field: $E_{x0}^{(i)} + E_{x0}^{(r)} = E_{x0}^{(a)} + E_{x0}^{(b)} \Rightarrow 1 + r = a + b.$ ← Eq. (I)

H-field: $H_{y0}^{(i)} + H_{y0}^{(r)} = H_{y0}^{(a)} + H_{y0}^{(b)} \Rightarrow -\frac{E_{x0}^{(i)}}{Z_0\cos\theta} + \frac{rE_{x0}^{(i)}}{Z_0\cos\theta} = -\frac{n_0^2 a E_{x0}^{(i)}}{Z_0\sqrt{n_0^2 - \sin^2\theta}} + \frac{n_0^2 b E_{x0}^{(i)}}{Z_0\sqrt{n_0^2 - \sin^2\theta}}$

$$\Rightarrow \frac{\sqrt{n_0^2 - \sin^2\theta}}{n_0^2\cos\theta}(1 - r) = a - b. \quad \leftarrow \text{Eq. (II)}$$

Continuity of E_x and H_y at $z=-d$:

E-field: $E_{x0}^{(a)} e^{-ik_z^{(a)}d} + E_{x0}^{(b)} e^{-ik_z^{(b)}d} = E_{x0}^{(t)} e^{-ik_z^{(t)}d} \Rightarrow$

$$a e^{+i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d} + b e^{-i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d} = \tau e^{+i(\omega/c)\sqrt{n_s^2 - \sin^2\theta}\,d} \quad \leftarrow \text{Eq. (III)}$$

H-field: $-\frac{n_0^2 a E_{x0}^{(i)}}{Z_0\sqrt{n_0^2 - \sin^2\theta}} e^{+i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d} + \frac{n_0^2 b E_{x0}^{(i)}}{Z_0\sqrt{n_0^2 - \sin^2\theta}} e^{-i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d} = -\frac{n_s^2 \tau E_{x0}^{(i)}}{Z_0\sqrt{n_s^2 - \sin^2\theta}} e^{+i(\omega/c)\sqrt{n_s^2 - \sin^2\theta}\,d}$

$$\Rightarrow \frac{n_0^2\sqrt{n_s^2 - \sin^2\theta}}{n_s^2\sqrt{n_0^2 - \sin^2\theta}}\left(a e^{+i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d} - b e^{-i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\,d}\right) = \tau e^{+i(\omega/c)\sqrt{n_s^2 - \sin^2\theta}\,d}$$

← Eq. (IV)

From Eqs. (I) and (II) we solve for $\underline{a}$ and $\underline{b}$ in terms of $\underline{r}$. To simplify the notation, let us define $c = \dfrac{\sqrt{n_0^2 - \sin^2\theta}}{n_0^2 \cos\theta}$. Then:

$$\begin{cases} a + b = 1 + r \\ a - b = c(1-r) \end{cases} \Longrightarrow \begin{cases} a = \frac{1}{2}(1+c) + \frac{r}{2}(1-c) & \checkmark \\ b = \frac{1}{2}(1-c) + \frac{r}{2}(1+c) & \checkmark \end{cases}$$

In Eqs. (III) and (IV) the right-hand sides are the same; therefore, the left-hand sides must be equal. For simplicity we define $D = \dfrac{n_0^2\sqrt{n_s^2 - \sin^2\theta}}{n_s^2\sqrt{n_0^2 - \sin^2\theta}}$.
Then:

$$a e^{i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\, d} + b e^{-i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\, d} = D\left(a e^{i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\, d} - b e^{-i(\omega/c)\sqrt{n_0^2 - \sin^2\theta}\, d}\right)$$

$$\Longrightarrow (D-1)\, a\, e^{+2i(\omega/c)\, d\sqrt{n_0^2 - \sin^2\theta}} = (D+1)\, b \Longrightarrow$$

$$\left(\frac{D-1}{D+1}\right)\left[\frac{1}{2}(1+c) + \frac{r}{2}(1-c)\right] e^{2i(\omega/c)\, d\sqrt{n_0^2 - \sin^2\theta}} = \frac{1}{2}(1-c) + \frac{r}{2}(1+c) \Longrightarrow$$

$$\left(\frac{D-1}{D+1}\right)\left[1 + r\left(\frac{1-c}{1+c}\right)\right] e^{2i(\omega/c)\, d\sqrt{n_0^2 - \sin^2\theta}} = r + \left(\frac{1-c}{1+c}\right) \Longrightarrow$$

$$r\left[1 + \left(\frac{c-1}{c+1}\right)\left(\frac{D-1}{D+1}\right) e^{2i(\omega/c)\, d\sqrt{n_0^2 - \sin^2\theta}}\right] = \left(\frac{c-1}{c+1}\right) + \left(\frac{D-1}{D+1}\right) e^{i2(\omega/c)\, d\sqrt{n_0^2 - \sin^2\theta}}.$$

We now recognize $\dfrac{c-1}{c+1} = \dfrac{\sqrt{n_0^2 - \sin^2\theta} - n_0^2\cos\theta}{\sqrt{n_0^2 - \sin^2\theta} + n_0^2\cos\theta}$ as the Fresnel reflection coefficient at the interface between the incidence medium and the dielectric coating layer. We shall denote this parameter as ρ_{p1}. Similarly,

$$\frac{D-1}{D+1} = \frac{n_0^2\sqrt{n_s^2 - \sin^2\theta} - n_s^2\sqrt{n_0^2 - \sin^2\theta}}{n_0^2\sqrt{n_s^2 - \sin^2\theta} + n_s^2\sqrt{n_0^2 - \sin^2\theta}}$$ is the Fresnel reflection coefficient at the interface between the coating layer and the substrate. This coefficient shall be denoted by ρ_{p2}. We will have:

$$r = \frac{\rho_{p_1} + \rho_{p_2} \exp\left[i2(\omega/c)d\sqrt{n_0^2 - \sin^2\theta}\,\right]}{1 + \rho_{p_1}\rho_{p_2} \exp\left[i2(\omega/c)d\sqrt{n_0^2 - \sin^2\theta}\,\right]}.$$

Having found r, it is now easy to find a, b and c from the preceding equations. The results can be readily used when n_0 and/or n_s are complex-valued, provided that attention is paid to the signs of the square roots. The same procedure as used in this problem may be applied to the case of s-polarized light as well.

The solution for multilayer problems containing an arbitrary number of metal and dielectric layers follows exactly the same procedure. In the end, however, finding analytical results for multilayer stacks containing two or more layers is cumbersome, and numerical solutions of the equations are preferred.