Problem 4)

a) $\dfrac{\partial \mathcal{E}(\vec{r},t)}{\partial t} = [\vec{E}'(\vec{r})\cos\omega t + \vec{E}''(\vec{r})\sin\omega t]\cdot\dfrac{\partial}{\partial t}[\epsilon_0|\epsilon(\omega)|\vec{E}'(\vec{r})\cos(\omega t-\phi_\epsilon) +$

$\epsilon_0|\epsilon(\omega)|\vec{E}''(\vec{r})\sin(\omega t-\phi_\epsilon)] + [\vec{H}'(\vec{r})\cos\omega t + \vec{H}''(\vec{r})\sin\omega t]\cdot\dfrac{\partial}{\partial t}[\mu_0|\mu(\omega)|\vec{H}'(\vec{r})\cos(\omega t-\phi_\mu)$

$+\mu_0|\mu(\omega)|\vec{H}''(\vec{r})\sin(\omega t-\phi_\mu)]$

$= \epsilon_0|\epsilon(\omega)|\{-\omega E'^2(\vec{r})\cos\omega t\sin(\omega t-\phi_\epsilon) + \omega E''^2(\vec{r})\sin\omega t\cos(\omega t-\phi_\epsilon) + \omega\vec{E}'(\vec{r})\cdot\vec{E}''(\vec{r})\cos\omega t\cos(\omega t-\phi_\epsilon)$

$-\omega\vec{E}'(\vec{r})\cdot\vec{E}''(\vec{r})\sin\omega t\sin(\omega t-\phi_\epsilon)\} + \mu_0|\mu(\omega)|\{-\omega H'^2(\vec{r})\cos\omega t\sin(\omega t-\phi_\mu)$

$+\omega H''^2(\vec{r})\sin\omega t\cos(\omega t-\phi_\mu) + \omega\vec{H}'(\vec{r})\cdot\vec{H}''(\vec{r})\cos\omega t\cos(\omega t-\phi_\mu) - \omega\vec{H}'(\vec{r})\cdot\vec{H}''(\vec{r})\sin\omega t\sin(\omega t-\phi_\mu)\}$

$= \epsilon_0|\epsilon(\omega)|\omega\{-\frac{1}{2}E'^2(\vec{r})\cos\phi_\epsilon\sin 2\omega t + E'^2(\vec{r})\sin\phi_\epsilon\cos^2\omega t + \frac{1}{2}E''^2(\vec{r})\cos\phi_\epsilon\sin 2\omega t$

$+ E''^2(\vec{r})\sin\phi_\epsilon\sin^2\omega t + \vec{E}'(\vec{r})\cdot\vec{E}''(\vec{r})\cos(2\omega t-\phi_\epsilon)\}$

$+\mu_0|\mu(\omega)|\omega\{-\frac{1}{2}H'^2(\vec{r})\cos\phi_\mu\sin 2\omega t + H'^2(\vec{r})\sin\phi_\mu\cos^2\omega t + \frac{1}{2}H''^2(\vec{r})\cos\phi_\mu\sin 2\omega t$

$+ H''^2(\vec{r})\sin\phi_\mu\sin^2\omega t + \vec{H}'(\vec{r})\cdot\vec{H}''(\vec{r})\cos(2\omega t-\phi_\mu)\}$

$= \epsilon_0|\epsilon(\omega)|\omega\{\frac{1}{2}[E'^2(\vec{r})+E''^2(\vec{r})]\sin\phi_\epsilon - \frac{1}{2}[E'^2(\vec{r})-E''^2(\vec{r})]\sin(2\omega t-\phi_\epsilon) + \vec{E}'(\vec{r})\cdot\vec{E}''(\vec{r})\cos(2\omega t-\phi_\epsilon)\}$

$+\mu_0|\mu(\omega)|\omega\{\frac{1}{2}[H'^2(\vec{r})+H''^2(\vec{r})]\sin\phi_\mu - \frac{1}{2}[H'^2(\vec{r})-H''^2(\vec{r})]\sin(2\omega t-\phi_\mu) + \vec{H}'(\vec{r})\cdot\vec{H}''(\vec{r})\cos(2\omega t-\phi_\mu)\}$

$$\Rightarrow \left\langle \frac{\partial}{\partial t}\mathcal{E}(\vec{r},t)\right\rangle = \frac{\omega}{2}\epsilon_0|\epsilon(\omega)|\sin\phi_\epsilon(\omega)[E'^2(\vec{r})+E''^2(\vec{r})] + \frac{\omega}{2}\mu_0|\mu(\omega)|\sin\phi_\mu(\omega)[H'^2(\vec{r})+H''^2(\vec{r})]$$

- When $\sin\phi_\epsilon = 0$, i.e., $\phi_\epsilon = 0$ or π, <u>no</u> energy is given or taken from the <u>electric dipoles</u> (on average). In this case there is <u>neither</u> absorption nor gain.
- When $\sin\phi_\epsilon > 0$, i.e., $0 < \phi_\epsilon < \pi$, on average some energy goes into the dipoles. In this case there is <u>absorption</u>.
- When $\sin\phi_\epsilon < 0$, i.e., $\pi < \phi_\epsilon < 2\pi$, on average some energy comes out of the dipoles. In this case there is <u>optical gain</u>.

The same statements can be made about Φ_μ, with energy going into or coming out of the magnetic dipoles in this case.

b) $\vec{S}(\vec{r},t) = \mathrm{Re}\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\} \times \mathrm{Re}\{\vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}$

$$= \{\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)+\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)\}\times\{\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)+\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)\}$$

$$= (\vec{E}_0'\times\vec{H}_0')\cos^2(\vec{k}'\cdot\vec{r}-\omega t)+(\vec{E}_0''\times\vec{H}_0'')\sin^2(\vec{k}'\cdot\vec{r}-\omega t)+$$

$$\frac{1}{2}(\vec{E}_0'\times\vec{H}_0''+\vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}'\cdot\vec{r}-\omega t)] \Rightarrow$$

$$\vec{S}(\vec{r},t)=\frac{1}{2}(\vec{E}_0'\times\vec{H}_0'+\vec{E}_0''\times\vec{H}_0'')+\frac{1}{2}(\vec{E}_0'\times\vec{H}_0'-\vec{E}_0''\times\vec{H}_0'')\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]+\frac{1}{2}(\vec{E}_0'\times\vec{H}_0''+\vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]$$

Now, Maxwell's 3rd equation yields: $\vec{k}'\times\vec{E}_0=\omega\mu_0\mu\vec{H}_0 \Rightarrow$ $\vec{H}_0'=(\vec{k}'\times\vec{E}_0')/(\mu_0\mu\omega)$ and $\vec{H}_0''=(\vec{k}'\times\vec{E}_0'')/(\mu_0\mu\omega)$, because $\vec{k}$ and $\mu(\omega)$ are both real-valued. Also, from Maxwell's 1st equation we have: $\vec{k}'\cdot\vec{E}_0=0 \Rightarrow \vec{k}'\cdot\vec{E}_0'=0$ and $\vec{k}'\cdot\vec{E}_0''=0$. The Poynting vector becomes:

$$\vec{S}(\vec{r},t)=\frac{1}{2\mu_0\mu\omega}\Big\{\vec{E}_0'\times(\vec{k}'\times\vec{E}_0')+\vec{E}_0''\times(\vec{k}'\times\vec{E}_0'')+[\vec{E}_0'\times(\vec{k}'\times\vec{E}_0')-\vec{E}_0''\times(\vec{k}'\times\vec{E}_0'')]\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]$$

$$+[\vec{E}_0'\times(\vec{k}'\times\vec{E}_0'')+\vec{E}_0''\times(\vec{k}'\times\vec{E}_0')]\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]\Big\} \Rightarrow$$

$$\vec{S}(\vec{r},t)=\frac{1}{2\mu_0\mu\omega}\Big\{(E_0'^2+E_0''^2)\vec{k}'+(E_0'^2-E_0''^2)\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]\vec{k}'+2\vec{E}_0'\cdot\vec{E}_0''\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]\vec{k}'\Big\}$$

The direction of flow of energy is seen to be $\vec{k}'$. Next, we calculate $\vec{\nabla}\cdot\vec{S}$ as follows:

$$\vec{\nabla}\cdot\vec{S}(\vec{r},t)=\frac{1}{2\mu_0\mu\omega}\Big\{-2(\vec{k}'\cdot\vec{k}')(E_0'^2-E_0''^2)\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]+4(\vec{k}'\cdot\vec{k}')\vec{E}_0'\cdot\vec{E}_0''\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]\Big\}$$

We know that in homogeneous, isotropic, linear media $\vec{k}'\cdot\vec{k}'=(\omega/c)^2\mu(\omega)\epsilon(\omega)$ in the absence of $\vec{k}''$. Therefore:

$$\vec{\nabla}\cdot\vec{S}(\vec{r},t)=-\epsilon_0\epsilon\omega\Big\{+(E_0'^2-E_0''^2)\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]+2\vec{E}_0'\cdot\vec{E}_0''\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]\Big\}$$

The time-rate-of-change of local energy density is given by:

$$\frac{\partial}{\partial t}\mathcal{E}(\vec{r},t) = [\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]\cdot\frac{\partial}{\partial t}\{\epsilon_0\epsilon(\omega)[\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]\}$$

$$+[\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]\cdot\frac{\partial}{\partial t}\{\mu_0\mu(\omega)[\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]\}$$

$$=\frac{1}{2}\epsilon_0\epsilon(\omega)\frac{\partial}{\partial t}[\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2+\frac{1}{2}\mu_0\mu(\omega)\frac{\partial}{\partial t}[\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t-\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2$$

The local energy density may thus be written as:

$$\mathcal{E}(\vec{r},t)=\frac{1}{2}\epsilon_0\epsilon(\omega)[\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)+\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2+\frac{1}{2}\mu_0\mu(\omega)[\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2$$

$$=\frac{1}{2}\epsilon_0\epsilon(\omega)\{E_0'^2\cos^2(\vec{k}'\cdot\vec{r}-\omega t)+E_0''^2\sin^2(\vec{k}'\cdot\vec{r}-\omega t)+\vec{E}_0'\cdot\vec{E}_0''\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]\}$$

$$+\frac{1}{2}\mu_0\mu(\omega)\{\vec{H}_0'^2\cos^2(\vec{k}'\cdot\vec{r}-\omega t)+H_0''^2\sin^2(\vec{k}'\cdot\vec{r}-\omega t)+\vec{H}_0'\cdot\vec{H}_0''\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]\}$$

Now, $H_0'^2=\vec{H}_0'\cdot\vec{H}_0'=\frac{1}{\omega\mu_0\mu}(\vec{k}'\times\vec{E}_0')\cdot\vec{H}_0'=-\frac{1}{\omega\mu_0\mu}(\vec{k}'\times\vec{H}_0')\cdot\vec{E}_0'=\frac{\omega\epsilon_0\epsilon}{\omega\mu_0\mu}\vec{E}_0'\cdot\vec{E}_0'$.

Similarly, $H_0''^2=\frac{\epsilon_0\epsilon}{\mu_0\mu}\vec{E}_0''\cdot\vec{E}_0''$ and $\vec{H}_0'\cdot\vec{H}_0''=\frac{\epsilon_0\epsilon}{\mu_0\mu}\vec{E}_0'\cdot\vec{E}_0''$. Consequently:

$$\mathcal{E}(\vec{r},t)=\epsilon_0\epsilon(\omega)[\vec{E}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)+\vec{E}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2=\mu_0\mu(\omega)[\vec{H}_0'\cos(\vec{k}'\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}'\cdot\vec{r}-\omega t)]^2$$

In other words, the energy densities of the E-field and the H-field are equal.

$$\frac{\partial}{\partial t}\mathcal{E}(\vec{r},t)=\epsilon_0\epsilon(\omega)\{\omega E_0'^2\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]-\omega E_0''^2\sin[2(\vec{k}'\cdot\vec{r}-\omega t)]+2\omega\vec{E}_0'\cdot\vec{E}_0''\cos[2(\vec{k}'\cdot\vec{r}-\omega t)]\}$$

Clearly $\vec{\nabla}\cdot\vec{S}(\vec{r},t)+\frac{\partial}{\partial t}\mathcal{E}(\vec{r},t)=0$ and, therefore, the energy continuity equation is satisfied.