Problem 1)

a) $\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{\nabla}\times H(\vec{r})e^{-i\omega t} = \sigma(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t} - i\omega\epsilon_0\epsilon(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}$

$\Rightarrow \vec{\nabla}\times\vec{H}(\vec{r}) = [\sigma(\vec{r},\omega) - i\omega\epsilon_0\epsilon(\vec{r},\omega)]\vec{E}(\vec{r}) = -i\omega\epsilon_0\left[\epsilon(\vec{r},\omega) + i\frac{\sigma(\vec{r},\omega)}{\epsilon_0\omega}\right]\vec{E}(\vec{r}) = -i\omega\epsilon_0\epsilon_{eff}(\vec{r},\omega)\vec{E}(\vec{r})$

where $\epsilon_{eff}(\vec{r},\omega) = \epsilon(\vec{r},\omega) + i\frac{\sigma(\vec{r},\omega)}{\epsilon_0\omega}$.

In general, both $\epsilon(\vec{r},\omega)$, the dielectric function associated with bound electrons, and $\sigma(\vec{r},\omega)$, the electrical conductivity associated with conduction electrons, are complex functions of $\vec{r}$ and ω. The effective dielectric function $\epsilon_{eff}(\vec{r},\omega)$ is, therefore, a complex function that incorporates the contributions of both bound electrons and conduction electrons.

Metals and dielectrics can therefore be treated in the same way, when monochromatic electromagnetic waves propagate through them. Whether the electrons in these media are bound electrons or free electrons residing in the conduction band, all one needs to know is the effective dielectric function $\epsilon_{eff}(\vec{r},\omega)$. There is no need to distinguish the conductivity $\sigma(\vec{r},\omega)$ from the bound-electron permittivity $\epsilon(\vec{r},\omega)$. The only thing that matters is their combination in the form of $\epsilon_{eff}(\vec{r},\omega)$.

---

b) We must show that the results of part (a) are consistent with other Maxwell's equations as well. The only other equation that is relevant here is equation (1) of Maxwell. From the continuity equation of charge we have:

$\vec{\nabla}\cdot\vec{J}_{free} + \frac{\partial \rho_{free}}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J}_{free} - i\omega\rho_{free} = 0 \Rightarrow \vec{\nabla}\cdot[\sigma(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}] = i\omega\rho_{free}(\vec{r},t)$.

Maxwell's 1st equation: $\vec{\nabla}\cdot\vec{D} = \rho_{free} \Rightarrow \vec{\nabla}\cdot[\epsilon_0\epsilon(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}] = \frac{1}{i\omega}\vec{\nabla}\cdot[\sigma(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}]$

$\Rightarrow \vec{\nabla}\cdot\left[\epsilon_0\epsilon(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t} - \frac{1}{i\omega}\sigma(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}\right] = 0 \Rightarrow \vec{\nabla}\cdot[\epsilon_0\epsilon_{eff}(\vec{r},\omega)\vec{E}(\vec{r})e^{-i\omega t}] = 0$ ✓