Problem 6.14)

a) $\boldsymbol{p}(t) = -qx(t)\hat{\boldsymbol{x}}$. Therefore,

$$\frac{d\mathcal{E}(t)}{dt} = \boldsymbol{E}(t) \cdot \frac{d\boldsymbol{p}(t)}{dt} = q\omega|x_0|E_{x0}\cos(\omega t)\sin(\omega t - \varphi_0). \quad (1)$$

b) $$\frac{d\mathcal{E}_K(t)}{dt} = mv_x(t)\frac{dv_x(t)}{dt} = m\left[\frac{dx(t)}{dt}\right]\left[\frac{d^2x(t)}{dt^2}\right] = m\omega^3|x_0|^2\sin(\omega t - \varphi_0)\cos(\omega t - \varphi_0). \quad (2)$$

c) $$\frac{d\mathcal{E}_P(t)}{dt} = \alpha x(t)\frac{dx(t)}{dt} = -\alpha\omega|x_0|^2\cos(\omega t - \varphi_0)\sin(\omega t - \varphi_0). \quad (3)$$

d) $$\frac{d\mathcal{E}_L(t)}{dt} = \beta\left[\frac{dx(t)}{dt}\right]^2 = \beta\omega^2|x_0|^2\sin^2(\omega t - \varphi_0). \quad (4)$$

e) $$\begin{aligned}\frac{d\mathcal{E}_{\text{total}}(t)}{dt} &= \frac{d}{dt}[\mathcal{E}_K(t) + \mathcal{E}_P(t) + \mathcal{E}_L(t)] \\ &= \omega|x_0|^2\sin(\omega t - \varphi_0)\,[m\omega^2\cos(\omega t - \varphi_0) - \alpha\cos(\omega t - \varphi_0) + \beta\omega\sin(\omega t - \varphi_0)] \\ &= m\omega|x_0|^2\sin(\omega t - \varphi_0)\,[(\omega^2 - \omega_0^2)\cos(\omega t - \varphi_0) + \gamma\omega\sin(\omega t - \varphi_0)]. \quad (5)\end{aligned}$$

Now, according to the Lorentz oscillator model, we have

$$|x_0|\exp(\mathrm{i}\varphi_0) = \frac{(q/m)E_{x0}}{\omega^2 - \omega_0^2 + \mathrm{i}\gamma\omega} = \left(\frac{qE_{x0}}{m}\right)\frac{(\omega^2 - \omega_0^2) - \mathrm{i}\gamma\omega}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}. \quad (6)$$

Consequently,

$$|x_0| = (qE_{x0}/m)/\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}\,; \quad (7)$$

$$\cos(\varphi_0) = (\omega^2 - \omega_0^2)/\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}\,; \quad (8)$$

$$\sin(\varphi_0) = -\gamma\omega/\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}. \quad (9)$$

Equation (5) may now be rewritten, as follows:

$$\begin{aligned}\frac{d\mathcal{E}_{\text{total}}(t)}{dt} &= m\omega|x_0|^2\sin(\omega t - \varphi_0) \\ &\quad\times\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}\,\overbrace{[\cos(\varphi_0)\cos(\omega t - \varphi_0) - \sin(\varphi_0)\sin(\omega t - \varphi_0)]}^{\cos a\cos b - \sin a\sin b = \cos(a+b)} \\ &= q\omega|x_0|E_{x0}\cos(\omega t)\sin(\omega t - \varphi_0). \quad (10)\end{aligned}$$

A comparison of Eq.(10) with Eq.(1) reveals that $d\mathcal{E}(t)/dt = d[\mathcal{E}_K(t) + \mathcal{E}_P(t) + \mathcal{E}_L(t)]/dt$.