Problem 13) We solve Maxwell's equations for a uniformly-polarized spherical dipole of radius R and polarization $P_o\hat{\boldsymbol{z}}$, namely, $\boldsymbol{P}(\boldsymbol{r},t) = P_o\,\text{Sphere}(r/R)\exp(\omega''t)\cos(\omega't)\hat{\boldsymbol{z}}$, in the limit when $\omega'' \to 0$. The complex oscillation frequency, $\omega = \omega' \pm \mathrm{i}\omega''$, where ω'' is a small positive number, will eventually approach a real-valued frequency, that is, $\omega \to \omega'$.

We shall find exact expressions for the potentials $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$, and also for the radiated E- and H-fields, both inside and outside the spherical dipole. When the radius R of the sphere is small compared to the radiation wavelength $\lambda = 2\pi c/\omega$, the E-field inside the dipole may be expanded in a Taylor series with only the first few terms retained. One of these terms contributes to the spring-constant of the Lorentz oscillator model and accounts for the Clausius-Mossotti correction – a correction that is invoked when the Lorentz model is used in conjunction with dense aggregates of atoms and/or molecules. Another term represents a small correction to the mass m of the oscillating electron. The most important contribution to the internal E-field, however, comes from a 3rd-order term that opposes the oscillations and is, therefore, responsible for radiation resistance. We explore the consequences of this 3rd-order term for the behavior of the dipole, paying particular attention to damped oscillations that resemble spontaneous emission from atoms or molecules.

Fourier transform of spherical dipole. To find the scalar and vector potentials of an oscillating dipole using the Fourier transform method, we must allow for a small imaginary component ω'' of the oscillation frequency, otherwise the integrals will not converge. Defining the complex frequency $\omega = \omega' \pm \mathrm{i}\omega''$, where ω'' is a small positive number that will eventually approach zero, we write

$$\boldsymbol{P}(\boldsymbol{r},t) = \tfrac{1}{2}P_o\,\text{Sphere}(r/R)\{\exp[-\mathrm{i}(\omega'+\mathrm{i}\omega'')t] + \exp[\mathrm{i}(\omega'-\mathrm{i}\omega'')t]\}\hat{\boldsymbol{z}}. \qquad (1)$$

The spatial Fourier transform of $\boldsymbol{P}(\boldsymbol{r},t)$ is now evaluated as follows:

$$\boldsymbol{P}(\boldsymbol{k},t) = \tfrac{1}{2}P_o\int_{r=0}^{R}\int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\theta\,\mathrm{d}r\,[\exp(-\mathrm{i}\omega t) + \exp(\mathrm{i}\omega^* t)]\hat{\boldsymbol{z}}$$

$$= \frac{2\pi P_o}{k}\int_0^R r\sin(kr)\,\mathrm{d}r\,[\exp(-\mathrm{i}\omega t) + \exp(\mathrm{i}\omega^* t)]\hat{\boldsymbol{z}}$$

$$\boldsymbol{P}(\boldsymbol{k},t) = \frac{2\pi P_o(\sin kR - kR\cos kR)}{k^3}[\exp(-\mathrm{i}\omega t) + \exp(\mathrm{i}\omega^* t)]\hat{\boldsymbol{z}}. \qquad (2)$$

Calculating the vector potential. The vector potential for the first component of $\boldsymbol{P}(\boldsymbol{r},t)$, i.e., the component with frequency ω, is found as follows:

$$\boldsymbol{A}_1(\boldsymbol{r},t)=\frac{\mu_o}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{(-\mathrm{i}\omega)\boldsymbol{P}(\boldsymbol{k},t)}{k^2-(\omega/c)^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$=\frac{-\mathrm{i}\mu_o\omega P_o\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{z}}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\sin kR-kR\cos kR}{k^3[k^2-(\omega/c)^2]}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$=\frac{-\mathrm{i}\mu_o\omega P_o\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{z}}}{\pi r}\int_0^{\infty}\frac{(\sin kR-kR\cos kR)\sin(kr)}{k^2[k^2-(\omega/c)^2]}\mathrm{d}k$$

$$=\frac{-\mathrm{i}\mu_o\omega P_o\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{z}}}{2\pi r}\int_0^{\infty}\frac{[\cos k(r-R)-\cos k(r+R)]-kR[\sin k(r-R)+\sin k(r+R)]}{k^2[k^2+(-\mathrm{i}\omega/c)^2]}\mathrm{d}k$$

The integrals can be evaluated, and the final result is found to be

$$\boldsymbol{A}_1(\boldsymbol{r},t)=\frac{-\mu_o\omega P_o\hat{\boldsymbol{z}}}{4r(\omega/c)^3}\begin{cases}\left\{\begin{array}{l}\exp\{-\mathrm{i}\omega[t-(r+R)/c]\}-\exp\{-\mathrm{i}\omega[t-(r-R)/c]\}\\ -R(\mathrm{i}\omega/c)\{\exp\{-\mathrm{i}\omega[t-(r+R)/c]\}+\exp\{-\mathrm{i}\omega[t-(r-R)/c]\}\}\end{array}\right\}; & r>R\\ \left\{\begin{array}{l}-2r(\mathrm{i}\omega/c)\exp(-\mathrm{i}\omega t)+\exp\{-\mathrm{i}\omega[t-(R+r)/c]\}-\exp\{-\mathrm{i}\omega[t-(R-r)/c]\}\\ -R(\mathrm{i}\omega/c)\{\exp\{-\mathrm{i}\omega[t-(R+r)/c]\}-\exp\{-\mathrm{i}\omega[t-(R-r)/c]\}\}\end{array}\right\}; & r<R\end{cases}$$

Similarly, the vector potential for the second component with frequency ω^* is found to be

$$\boldsymbol{A}_2(\boldsymbol{r},t)=\frac{\mu_o}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\mathrm{i}\omega^*\boldsymbol{P}(\boldsymbol{k},t)}{k^2-(\omega^*/c)^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$=\frac{\mathrm{i}\mu_o\omega^*P_o\exp(\mathrm{i}\omega^*t)\hat{\boldsymbol{z}}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\sin kR-kR\cos kR}{k^3[k^2-(\omega^*/c)^2]}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$=\frac{\mathrm{i}\mu_o\omega^*P_o\exp(\mathrm{i}\omega^*t)\hat{\boldsymbol{z}}}{\pi r}\int_0^{\infty}\frac{(\sin kR-kR\cos kR)\sin(kr)}{k^2[k^2-(\omega^*/c)^2]}\mathrm{d}k$$

$$=\frac{\mathrm{i}\mu_o\omega^*P_o\exp(\mathrm{i}\omega^*t)\hat{\boldsymbol{z}}}{2\pi r}\int_0^{\infty}\frac{[\cos k(r-R)-\cos k(r+R)]-kR[\sin k(r-R)+\sin k(r+R)]}{k^2[k^2+(\mathrm{i}\omega^*/c)^2]}\mathrm{d}k$$

Once again, the integrals can be evaluated and the final result turns out to be

$$\boldsymbol{A}_2(\boldsymbol{r},t)=\frac{-\mu_o\omega^*P_o\hat{\boldsymbol{z}}}{4r(\omega^*/c)^3}\begin{cases}\left\{\begin{array}{l}\exp\{\mathrm{i}\omega^*[t-(r+R)/c]\}-\exp\{\mathrm{i}\omega^*[t-(r-R)/c]\}\\ +R(\mathrm{i}\omega^*/c)\{\exp\{\mathrm{i}\omega^*[t-(r+R)/c]\}+\exp\{\mathrm{i}\omega^*[t-(r-R)/c]\}\}\end{array}\right\}; & r>R\\ \left\{\begin{array}{l}+2r(\mathrm{i}\omega^*/c)\exp(\mathrm{i}\omega^*t)+\exp\{\mathrm{i}\omega^*[t-(R+r)/c]\}-\exp\{\mathrm{i}\omega^*[t-(R-r)/c]\}\\ +R(\mathrm{i}\omega^*/c)\{\exp\{\mathrm{i}\omega^*[t-(R+r)/c]\}-\exp\{\mathrm{i}\omega^*[t-(R-r)/c]\}\}\end{array}\right\}; & r<R\end{cases}$$

Adding $\boldsymbol{A}_1(\boldsymbol{r},t)$ and $\boldsymbol{A}_2(\boldsymbol{r},t)$ and letting ω^* approach ω, we find

$$\boldsymbol{A}(\boldsymbol{r},t)=-\frac{\mu_o\omega P_o\hat{\boldsymbol{z}}}{2r(\omega/c)^3}\begin{cases}\left\{\begin{array}{l}\cos\{\omega[t-(r+R)/c]\}-\cos\{\omega[t-(r-R)/c]\}\\ -R(\omega/c)\{\sin\{\omega[t-(r+R)/c]\}+\sin\{\omega[t-(r-R)/c]\}\}\end{array}\right\}; & r>R\\ \left\{\begin{array}{l}-2r(\omega/c)\sin(\omega t)+\cos\{\omega[t-(R+r)/c]\}-\cos\{\omega[t-(R-r)/c]\}\\ -R(\omega/c)\{\sin\{\omega[t-(R+r)/c]\}-\sin\{\omega[t-(R-r)/c]\}\}\end{array}\right\}; & r<R\end{cases}$$

$$= -\frac{\mu_o \omega P_o \hat{\boldsymbol{z}}}{r(\omega/c)^3}\begin{cases}[\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\sin[\omega(t-r/c)]; & r \ge R\\ \left\{\begin{array}{l} -(r\omega/c)\sin(\omega t) + \sin[\omega(t-R/c)]\sin(r\omega/c)\\ \quad +(R\omega/c)\sin(r\omega/c)\cos[\omega(t-R/c)]\end{array}\right\}; & r \le R\end{cases}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = -\frac{\mu_o \omega P_o \hat{\boldsymbol{z}}}{r(\omega/c)^3}\begin{cases}[\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\sin[\omega(t-r/c)]; & r \ge R\\ \{[\cos(R\omega/c) + (R\omega/c)\sin(R\omega/c)]\sin(r\omega/c) - (r\omega/c)\}\sin(\omega t) & \\ \quad -[\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\sin(r\omega/c)\cos(\omega t); & r \le R\end{cases} \tag{3}$$

Calculating the scalar potential. In the next step we calculate the scalar potential $\psi(\boldsymbol{r},t)$ for the spherical dipole whose Fourier transform $\boldsymbol{P}(\boldsymbol{k},t)$ is given in Eq.(2).

$$\begin{aligned}
\psi_1(\boldsymbol{r},t) &= \frac{1}{(2\pi)^3\varepsilon_o}\int_{-\infty}^{\infty}\frac{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},t)}{k^2-(\omega/c)^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}\\
&= \frac{-P_o\exp(-\mathrm{i}\omega t)}{(2\pi)^2\varepsilon_o}\int_{-\infty}^{\infty}\frac{\mathrm{i}k_z(\sin kR - kR\cos kR)}{k^3[k^2-(\omega/c)^2]}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}\\
&= \frac{-P_o\exp(-\mathrm{i}\omega t)}{(2\pi)^2\varepsilon_o}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\frac{(\sin kR - kR\cos kR)}{k^3[k^2-(\omega/c)^2]}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}\\
&= \frac{-P_o\exp(-\mathrm{i}\omega t)}{\pi\varepsilon_o}\frac{\partial}{\partial z}\int_0^{\infty}\frac{(\sin kR - kR\cos kR)\sin(kr)}{k^2[k^2-(\omega/c)^2]\,r}\mathrm{d}k\\
&= \frac{P_o z\exp(-\mathrm{i}\omega t)}{\pi\varepsilon_o r^3}\int_0^{\infty}\frac{(\sin kR - kR\cos kR)(\sin kr - kr\cos kr)}{k^2[k^2-(\omega/c)^2]}\mathrm{d}k\\
&= \frac{P_o z\exp(-\mathrm{i}\omega t)}{2\pi\varepsilon_o r^3}\left\{\int_0^{\infty}\frac{[\cos k(r-R) - \cos k(r+R)]}{k^2[k^2+(-\mathrm{i}\omega/c)^2]}\mathrm{d}k + rR\int_0^{\infty}\frac{[\cos k(r-R) + \cos k(r+R)]}{k^2+(-\mathrm{i}\omega/c)^2}\mathrm{d}k\right.\\
&\qquad\left. + \int_0^{\infty}\frac{(r-R)\sin k(r-R) - (r+R)\sin k(r+R)}{k[k^2+(-\mathrm{i}\omega/c)^2]}\mathrm{d}k\right.
\end{aligned}$$

$$\psi_1(\boldsymbol{r},t) = \frac{P_o z}{4\varepsilon_o r^3}\begin{cases}\dfrac{-2R(\mathrm{i}\omega/c)\exp(-\mathrm{i}\omega t) + \exp\{-\mathrm{i}\omega[t-(r+R)/c]\} - \exp\{-\mathrm{i}\omega[t-(r-R)/c]\}}{(-\mathrm{i}\omega/c)^3}\\ +\dfrac{rR\exp\{-\mathrm{i}\omega[t-(r+R)/c]\} + rR\exp\{-\mathrm{i}\omega[t-(r-R)/c]\}}{(-\mathrm{i}\omega/c)}\\ \dfrac{-2R\exp(-\mathrm{i}\omega t) + (r+R)\exp\{-\mathrm{i}\omega[t-(r+R)/c]\} - (r-R)\exp\{-\mathrm{i}\omega[t-(r-R)/c]\}}{(-\mathrm{i}\omega/c)^2}; & r > R\\ \dfrac{-2r(\mathrm{i}\omega/c)\exp(-\mathrm{i}\omega t) + \exp\{-\mathrm{i}\omega[t-(R+r)/c]\} - \exp\{-\mathrm{i}\omega[t-(R-r)/c]\}}{(-\mathrm{i}\omega/c)^3}\\ +\dfrac{rR\exp\{-\mathrm{i}\omega[t-(R+r)/c]\} + rR\exp\{-\mathrm{i}\omega[t-(R-r)/c]\}}{(-\mathrm{i}\omega/c)}\\ \dfrac{-2r\exp(-\mathrm{i}\omega t) + (R+r)\exp\{-\mathrm{i}\omega[t-(R+r)/c]\} - (R-r)\exp\{-\mathrm{i}\omega[t-(R-r)/c]\}}{(-\mathrm{i}\omega/c)^2}; & r < R\end{cases}$$

$$\psi_1(\boldsymbol{r},t)=\frac{-\mathrm{i}P_\mathrm{o}z}{4\varepsilon_\mathrm{o}r^3(\omega/c)^3}\begin{cases}[1-(\omega/c)^2rR-\mathrm{i}(\omega/c)(r+R)]\exp\{-\mathrm{i}\omega[t-(r+R)/c]\}\\-[1+(\omega/c)^2rR-\mathrm{i}(\omega/c)(r-R)]\exp\{-\mathrm{i}\omega[t-(r-R)/c]\}; & r>R\\[1-(\omega/c)^2rR-\mathrm{i}(\omega/c)(R+r)]\exp\{-\mathrm{i}\omega[t-(R+r)/c]\}\\-[1+(\omega/c)^2rR-\mathrm{i}(\omega/c)(R-r)]\exp\{-\mathrm{i}\omega[t-(R-r)/c]\}; & r<R\end{cases}$$

The second term in the expression of the scalar potential is obtained by replacing $\mathrm{i}\omega$ with $-\mathrm{i}\omega^*$ in the above equation, namely,

$$\psi_2(\boldsymbol{r},t)=\frac{\mathrm{i}P_\mathrm{o}z}{4\varepsilon_\mathrm{o}r^3(\omega^*/c)^3}\begin{cases}[1-(\omega^*/c)^2rR+\mathrm{i}(\omega^*/c)(r+R)]\exp\{\mathrm{i}\omega^*[t-(r+R)/c]\}\\-[1+(\omega^*/c)^2rR+\mathrm{i}(\omega^*/c)(r-R)]\exp\{\mathrm{i}\omega^*[t-(r-R)/c]\}; & r>R\\[1-(\omega^*/c)^2rR+\mathrm{i}(\omega^*/c)(R+r)]\exp\{\mathrm{i}\omega^*[t-(R+r)/c]\}\\-[1+(\omega^*/c)^2rR+\mathrm{i}(\omega^*/c)(R-r)]\exp\{\mathrm{i}\omega^*[t-(R-r)/c]\}; & r<R\end{cases}$$

Adding $\psi_1(\boldsymbol{r},t)$ and $\psi_2(\boldsymbol{r},t)$ and letting ω^* approach ω, we now find

$$\psi(\boldsymbol{r},t)=\frac{P_\mathrm{o}z}{2\varepsilon_\mathrm{o}r^3(\omega/c)^3}\begin{cases}-[1-(\omega/c)^2rR]\sin\{\omega[t-(r+R)/c]\}-(\omega/c)(r+R)\cos\{\omega[t-(r+R)/c]\}\\+[1+(\omega/c)^2rR]\sin\{\omega[t-(r-R)/c]\}+(\omega/c)(r-R)\cos\{\omega[t-(r-R)/c]\}; & r>R\\-[1-(\omega/c)^2rR]\sin\{\omega[t-(R+r)/c]\}-(\omega/c)(R+r)\cos\{\omega[t-(R+r)/c]\}\\+[1+(\omega/c)^2rR]\sin\{\omega[t-(R-r)/c]\}+(\omega/c)(R-r)\cos\{\omega[t-(R-r)/c]\}; & r<R\end{cases}$$

$$=\frac{P_\mathrm{o}z}{\varepsilon_\mathrm{o}r^3(\omega/c)^3}\begin{cases}[\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)]\{\cos[\omega(t-r/c)]-(r\omega/c)\sin[\omega(t-r/c)]\}; & r\ge R\\[\sin(r\omega/c)-(r\omega/c)\cos(r\omega/c)]\{\cos[\omega(t-R/c)]-(R\omega/c)\sin[\omega(t-R/c)]\}; & r\le R\end{cases}$$

$$=\frac{P_\mathrm{o}\cos\theta}{\varepsilon_\mathrm{o}r^2}\begin{cases}\left[\dfrac{\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)}{(\omega/c)^3}\right]\{\cos[\omega(t-r/c)]-(r\omega/c)\sin[\omega(t-r/c)]\}; & r\ge R\\\left[\dfrac{\sin(r\omega/c)-(r\omega/c)\cos(r\omega/c)}{(\omega/c)^3}\right]\{\cos[\omega(t-R/c)]-(R\omega/c)\sin[\omega(t-R/c)]\}; & r\le R\end{cases}$$

$$\psi(\boldsymbol{r},t)=\begin{cases}\dfrac{P_\mathrm{o}\cos\theta}{\varepsilon_\mathrm{o}r^2}\left[\dfrac{\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)}{(\omega/c)^3}\right]\{\cos[\omega(t-r/c)]-(r\omega/c)\sin[\omega(t-r/c)]\}; & r\ge R\\\dfrac{P_\mathrm{o}z}{\varepsilon_\mathrm{o}}\left[\dfrac{\sin(r\omega/c)-(r\omega/c)\cos(r\omega/c)}{(r\omega/c)^3}\right]\left\{\begin{matrix}[\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)]\sin(\omega t)\\+[\cos(R\omega/c)+(R\omega/c)\sin(R\omega/c)]\cos(\omega t)\end{matrix}\right\}; & r\le R\end{cases}\qquad(4)$$

Electric and magnetic fields inside and outside the dipole. Having found the potentials $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$, we now calculate the E- and B-fields as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t$$

$$= \begin{cases} -\dfrac{P_o}{\varepsilon_o}[\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\{(r\omega/c)^{-1}\cos[\omega(t-r/c)]\sin\theta\hat{\boldsymbol{\theta}} \\ \quad +\{(r\omega/c)^{-2}\sin[\omega(t-r/c)] - (r\omega/c)^{-3}\cos[\omega(t-r/c)]\}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})\}; \qquad r > R \\ -\dfrac{P_o\hat{\boldsymbol{z}}}{\varepsilon_o}\cos(\omega t) + \dfrac{P_o}{\varepsilon_o}\left\{\dfrac{\sin(r\omega/c) - (r\omega/c)\cos(r\omega/c)}{(r\omega/c)^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) - \dfrac{\sin(r\omega/c)}{(r\omega/c)}\sin\theta\hat{\boldsymbol{\theta}}\right\} \\ \quad \times\{[\cos(R\omega/c) + (R\omega/c)\sin(R\omega/c)]\cos(\omega t) \\ \quad + [\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\sin(\omega t)\}; \qquad r < R \end{cases} \tag{5}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$$

$$= -\frac{\mu_o\omega P_o\sin\theta\,\hat{\boldsymbol{\phi}}}{r^2(\omega/c)^3}\begin{cases} [\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\{(r\omega/c)\cos[\omega(t-r/c)] + \sin[\omega(t-r/c)]\}; \\ \qquad r \ge R \\ [\sin(r\omega/c) - (r\omega/c)\cos(r\omega/c)] \\ \quad \times\{[\cos(R\omega/c) + (R\omega/c)\sin(R\omega/c)]\sin(\omega t) \\ \quad - [\sin(R\omega/c) - (R\omega/c)\cos(R\omega/c)]\cos(\omega t)\}; \qquad r \le R \end{cases} \tag{6}$$

Approximate expressions for *E*- and *B*-fields of small spherical dipole. It is now possible to make approximations to the above expressions using Taylor series expansions, as the radius R of the sphere is much smaller than a wavelength. We will have

$$\boldsymbol{E}(\boldsymbol{r},t) \approx \begin{cases} -\dfrac{P_o R^3[1 - \frac{1}{10}(R\omega/c)^2]}{3\varepsilon_o r}\Big\{(\omega/c)^2\cos[\omega(t-r/c)]\sin\theta\hat{\boldsymbol{\theta}} \\ \quad + \{r^{-1}(\omega/c)\sin[\omega(t-r/c)] - r^{-2}\cos[\omega(t-r/c)]\}(2\cos\boldsymbol{\theta}\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})\Big\}; \quad r > R \\ -\dfrac{P_o}{3\varepsilon_o}\Big\{[1 - (R\omega/c)^2 + \frac{1}{5}(r\omega/c)^2]\hat{\boldsymbol{z}}\cos(\omega t) - \frac{1}{5}(r\omega/c)^2\sin\theta\hat{\boldsymbol{\theta}}\cos(\omega t) \\ \quad - \frac{2}{3}(R\omega/c)^3\hat{\boldsymbol{z}}\sin(\omega t)\Big\}; \qquad r < R. \end{cases} \tag{7}$$

$$\boldsymbol{B}(\boldsymbol{r},t) \approx \begin{cases} -\dfrac{\mu_o\omega P_o(4\pi R^3/3)\sin\theta}{4\pi r}\Big\{(\omega/c)\cos[\omega(t-r/c)] + r^{-1}\sin[\omega(t-r/c)]\Big\}\hat{\boldsymbol{\phi}}; \quad r \ge R \\ -\frac{1}{3}\mu_o\omega P_o r\sin\theta\Big\{[1 + \frac{1}{2}(R\omega/c)^2]\sin(\omega t) - \frac{1}{3}(R\omega/c)^3\cos(\omega t)\Big\}\hat{\boldsymbol{\phi}}; \qquad r \le R \end{cases} \tag{8}$$

Radiated power. In this section we compute the time-averaged Poynting vector of the electromagnetic field radiated by the dipole oscillator in the region $r > R$. Using Eqs.(5), and (6), the exact result is found to be

$$<\boldsymbol{S}(\boldsymbol{r},t)> = <\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)> = \frac{\omega P_o^2 \sin^2\theta\,\hat{\boldsymbol{r}}}{2\varepsilon_o r^2(\omega/c)^3}[\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)]^2. \qquad (9)$$

Integration over a sphere of radius r yields the total (time-averaged) radiated power as follows:

$$\text{Total power} = \frac{4\pi\omega P_o^2}{3\varepsilon_o(\omega/c)^3}[\sin(R\omega/c)-(R\omega/c)\cos(R\omega/c)]^2$$

$$\approx \frac{[(4\pi R^3/3)P_o]^2\omega^4}{12\pi\varepsilon_o c^3} = \frac{\mu_o p_o^2\omega^4}{12\pi c}. \qquad (10)$$

The same result may be obtained by calculating the time-averaged value of $\boldsymbol{E}\cdot\partial\boldsymbol{p}/\partial t$, where $\boldsymbol{E} = \frac{2P_o}{9\varepsilon_o}(R\omega/c)^3\sin(\omega t)\hat{\boldsymbol{z}}$ is the E-field component inside the dipole that is in-phase with the dipolar current, $\partial\boldsymbol{p}/\partial t$, with $\boldsymbol{p}(t) = (4\pi R^3/3)P_o\cos(\omega t)\hat{\boldsymbol{z}}$; see Eq.(7). The internal E-field extracts energy from the dipole and delivers it to the radiated field. This is the essence of radiation resistance.

The Lorentz oscillator model. Several terms contribute to the internal E-field of an oscillating dipole, as given by Eq.(7). The largest term is $\boldsymbol{E}^{(1)}(\boldsymbol{r},t) = -(P_o/3\varepsilon_o)\cos(\omega t)\hat{\boldsymbol{z}}$. This field exerts a restoring force on the oscillating charge in proportion to the separation between the positive and negative charges that constitute the dipole. In doing so, it contributes to the spring-constant of the Lorentz oscillator model, which is encoded as the resonance frequency ω_o. If the dipole is driven by an external E-field, $E_o\cos(\omega t+\phi_o)\hat{\boldsymbol{z}}$, then there is no need to account for the above internal E-field as the spring constant of the dipole already contains its contribution. If, however, the dipole is embedded in a continuous medium, and the total E-field (i.e., external field plus the field radiated by all the dipoles of the medium, including the local dipole) is computed at the location of the dipole, then the computed local field must be augmented by $+(P_o/3\varepsilon_o)\cos(\omega t)\hat{\boldsymbol{z}}$ to avoid double-counting the contribution of $\boldsymbol{E}^{(1)}(\boldsymbol{r},t)$. This is the logic behind the Clausius-Mossotti correction (also known as the Lorenz-Lorentz correction). It is worth mentioning that the Clausius-Mossotti correction is generally applied to gases and dielectric solids or liquids, where bound charges contribute to the susceptibility of the medium. In the case of conduction electrons (where the Lorentz and Drude models coincide), the Clausius-Mossotti correction is never applied, the reason being that the spring-constant associated with conduction electrons is vanishingly small (i.e., $\omega_o = 0$), and, therefore, there is no need for the self-field correction.

The second contribution to the internal E-field of an oscillating dipole may be written as follows:

$$\boldsymbol{E}^{(2)}(\boldsymbol{r},t) = \tfrac{1}{3}\mu_o P_o R^2\omega^2[\hat{\boldsymbol{z}} - \tfrac{1}{5}(r/R)^2(\hat{\boldsymbol{z}} - \sin\theta\,\hat{\boldsymbol{\theta}})]\cos(\omega t). \qquad (11)$$

The spherical symmetry of $\boldsymbol{E}^{(2)}(\boldsymbol{r},t)$ ensures that its net contribution is along the z-axis, while its proportionality to ω^2 reveals its association with the second derivative of the displacement of the oscillating charge. The corresponding force, therefore, has the effect of a small correction to the mass m of the mobile charge q of the Lorentz oscillator model. The correction to m, which is on the order of $\mu_0 q^2/(4\pi R)$, where $q \sim 1.6\times10^{-19}$ C is the electronic charge and $R \sim 10^{-10}$ m is the dipole radius, is indeed several orders of magnitude smaller than the electron's mass. For a single dipole driven by an external field, this correction is already included in the Lorentz oscillator model (in the form of an *effective* m). For an embedded dipole within a continuum, the oscillator's mass should, in principle, be adjusted, since the computed local E-field does not separate $\boldsymbol{E}^{(2)}(\boldsymbol{r},t)$ from the E-field that drives the dipole. In practice, however, the correction is so small that it is always neglected.

The third and final contribution to the internal E-field is given by

$$\boldsymbol{E}^{(3)}(\boldsymbol{r},t) = \frac{\mu_o}{6\pi c}(4\pi R^3/3)P_o\,\omega^3\sin(\omega t)\hat{\boldsymbol{z}}. \tag{12}$$

This field, which is entirely responsible for the radiation resistance to dipole oscillations, exerts a braking force on the mobile charge q analogous to the friction (or damping) term $\gamma \mathrm{d}p(t)/\mathrm{d}t$ of the Lorentz oscillator model, whose governing equation is

$$\frac{\mathrm{d}^2\boldsymbol{p}(t)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \omega_o^2\boldsymbol{p}(t) = (q^2/m)\boldsymbol{E}_{\text{ext}}(t). \tag{13}$$

Writing $p_o = (4\pi R^3/3)P_o$ for the magnitude of the dipole moment, and noting that $p_o\omega\sin(\omega t)$ is the time derivative of $-p_o\cos(\omega t)$, which is proportional to the velocity of the mobile charge q, the damping coefficient associated with radiation resistance is seen to be $\gamma = \mu_o q^2\omega_o^2/(6\pi mc)$. [Note that the internal E-field, in much the same way as the external field, is multiplied by q^2/m to yield a term on the left-hand side of Eq.(13).] With $\mu_o = 4\pi\times10^{-7}$ henry/meter, $q = 1.6\times10^{-19}$ C, $m = 0.911\times10^{-30}$ Kg, and $c = 3\times10^8$ m/s, the numerical value of γ turns out to be $\Gamma\omega_o^2 = 6.25\times10^{-24}\omega_o^2$ hertz.

One problem with the above model is that the damping coefficient γ, which must be a constant if the linear system is to be time-invariant, varies with the oscillation frequency ω. Thus the damped oscillator cannot be modeled as a time-invariant system and, strictly speaking, Eq.(13) cannot be expected to hold for anything other than sinusoidal excitations of the dipole. Of course, if the range of oscillation frequencies involved in any particular excitation is sufficiently narrow, one may expect the differential equation to remain approximately valid provided that γ is set to a constant value by fixing ω at the center frequency of the oscillations. In particular, the impulse-response of the dipole, which typically has a narrow spectrum, may be obtained from Eq.(13) in the usual way. This impulse-response then represents the spontaneous emission of a photon when an atom goes from an excited state to a lower-energy state. The impulse-response is given by $p(t) = p_o\exp(-\tfrac{1}{2}\gamma t)\sin(\sqrt{\omega_o^2 - \tfrac{1}{4}\gamma^2}\,t)\,\text{Step}(t)$, with γ being fixed at $\Gamma\omega_o^2$, where $\Gamma = \mu_o q^2/(6\pi mc) \approx 6.25\times10^{-24}$ sec. The natural line-width is thus seen to be $\Delta\omega \sim \Gamma\omega_o^2$. Considering that $\lambda = 2\pi c/\omega$, we will have $\Delta\lambda \sim 2\pi c\Delta\omega/\omega_o^2 = 2\pi c\Gamma \approx 11.8\times10^{-15}$ m. This is the universal line-width of spontaneous emission according to the classical theory of electrodynamics.

Another problem with Eq.(13) is that its Fourier transform yields

$$p(\omega) = \frac{(q^2/m)E_{\text{ext}}(\omega)}{\omega_o^2 - \omega^2 - \mathrm{i}\gamma\omega}. \tag{14}$$

When the dipole is excited by a delta-function, $\boldsymbol{E}_{\text{ext}}(t) = \delta(t)\hat{\boldsymbol{z}}$, we will have $E_{\text{ext}}(\omega) = 1.0$, and, therefore,

$$|p(\omega)|^2 = \frac{(q^2/m)^2}{(\omega^2 - \omega_o^2)^2 + \gamma^2\omega^2}. \tag{15}$$

In accordance with Eq.(10), the rate of energy radiation (i.e., energy output per unit time) by an oscillating dipole $p_o\cos(\omega t)\hat{\boldsymbol{z}}$ is given by $\mu_o p_o^2\omega^4/(12\pi c)$. The total radiated energy of spontaneous emission must, therefore, be proportional to $\int_0^\infty |p(\omega)|^2\omega^4 \mathrm{d}\omega$,[1] which is a divergent integral. This indicates that the Lorentz model, which associates radiation resistance losses with the 1st derivative of dipole oscillations, is fundamentally flawed. A better way to incorporate the effects of radiation resistance into the Lorentz oscillator model is by recognizing the term in Eq.(7) that is proportional to $p_o\omega^3\sin(\omega t)$ as the 3rd derivative of $p_o\cos(\omega t)$, in which case the differential equation governing the dipole oscillations becomes

$$-\Gamma\frac{\mathrm{d}^3\boldsymbol{p}(t)}{\mathrm{d}t^3} + \frac{\mathrm{d}^2\boldsymbol{p}(t)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \underset{\sim}{\omega}_o^2\boldsymbol{p}(t) = (q^2/m)\boldsymbol{E}_{\text{ext}}(t). \tag{16}$$

We have included the term $\gamma\mathrm{d}\boldsymbol{p}(t)/\mathrm{d}t$ in the above equation to allow for damping mechanisms *other than* radiation resistance. Note, once again, that our small spherical dipole gives rise to a radiation resistance term that is distinct from the externally-applied excitation field $E_{\text{ext}}\cos(\omega t)$. In contrast, if the dipole happens to be embedded in a continuous medium, in which case the exciting field is the external field plus the collective field of all the dipoles in the medium (including the local dipole), then the radiation resistance term $-\Gamma\mathrm{d}^3\boldsymbol{p}(t)/\mathrm{d}t^3$ should be dropped from the Lorentz model of Eq.(16).

The differential equation incorporating the 3rd derivative of the oscillating dipole has its own stability problems. Considering the orders of magnitude of the parameters involved, the characteristic equation $-\Gamma s^3 + s^2 + \gamma s + \omega_o^2 = 0$ may be factored as $(1-\Gamma s)[s^2 + (\Gamma\omega_o^2 + \gamma)s + \omega_o^2] = 0$.[2] While the pair of complex-conjugate roots of this equation are the same as before, the new (positive) root, $s = 1/\Gamma$, causes exponential growth of the impulse-response (i.e., damped oscillations associated with spontaneous emission) and is, therefore, unphysical.

Including the next higher-order term in the expansion of the internal E-field of the dipole oscillator in Eq.(7) does not help with its unphysical behavior either. The 4th order contribution to the internal E-field is readily found to be

$$\boldsymbol{E}^{(4)}(\boldsymbol{r},t) = -\frac{\mu_o P_o R^4\omega^4}{12c^2}[\hat{\boldsymbol{z}} + \tfrac{2}{5}(r/R)^2(\hat{\boldsymbol{z}} - \sin\theta\hat{\boldsymbol{\theta}}) - \tfrac{1}{35}(r/R)^4(\hat{\boldsymbol{z}} - 2\sin\theta\hat{\boldsymbol{\theta}})]\cos(\omega t). \tag{17}$$

The spherical symmetry of the dipole ensures that the contribution to the Lorentz oscillator model of the spatially-averaged 4th order term in Eq.(17) is aligned with the z-axis. Defining the new parameter $\Omega = \mu_o R q^2/(16\pi m c^2) \sim 10^{-42}\ \text{sec}^2$, the differential equation governing the Lorentz oscillator model becomes

$$\Omega\frac{\mathrm{d}^4\boldsymbol{p}(t)}{\mathrm{d}t^4} - \Gamma\frac{\mathrm{d}^3\boldsymbol{p}(t)}{\mathrm{d}t^3} + \frac{\mathrm{d}^2\boldsymbol{p}(t)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} + \underset{\sim}{\omega}_o^2\boldsymbol{p}(t) = (q^2/m)\boldsymbol{E}_{\text{ext}}(t). \tag{18}$$

To a good approximation, the characteristic equation of the above differential equation may be factored as follows: $(\Omega s^2 - \Gamma s + 1)[s^2 + (\Gamma\omega_o^2 + \gamma)s + \omega_o^2] = 0$.[2] As before, spontaneous emission will exhibit a frequency $\omega_o\sqrt{1 - \tfrac{1}{4}(\Gamma\omega_o + \gamma\omega_o^{-1})^2}$ and a damping rate $\tfrac{1}{2}(\Gamma\omega_o^2 + \gamma)$, while the remaining roots produce the unstable oscillation $\exp[(\Gamma/2\Omega)t]\cos[\Omega^{-1/2}\sqrt{1-(\Gamma^2/4\Omega)}\ t]$. This unphysical solution must somehow be countered by the internal dynamics of the atom, as real atoms do not exhibit such exploding behavior during spontaneous emission.

For a phenomenological treatment of spontaneous emission, where the characteristic behavior of the standard (i.e., 2nd order) Lorentz oscillator model of Eq.(13) is retained while the transition to oscillations in the vicinity of $t=0$ is smoothed out to produce a finite amount of radiated energy, see Problem 16, Chapter 4.

Footnote 1: Assume $p(\omega)$ is sampled at intervals of $\Delta\omega$. The spontaneous emission is then the sum of many single-frequency oscillations, whose radiation rates are proportional to $|p(\omega_n)|^2(\Delta\omega)^2\omega_n^4$. The periodic signal obtained by a superposition of discrete frequencies has a period of $1/\Delta\omega$. Multiplying the radiation rate of each oscillator with the repetition period $1/\Delta\omega$ yields the total radiated energy of the oscillator. The total radiated energy is thus the sum over n of $|p(\omega_n)|^2\omega_n^4\Delta\omega$ which approaches $\int_0^\infty |p(\omega)|^2\omega^4 \mathrm{d}\omega$ in the limit when $\Delta\omega\rightarrow 0$.

Footnote 2: The 3rd and 4th order characteristic equations of the Lorentz oscillator model have been factored out approximately. Considering the orders of magnitude of the parameters involved, these approximations are justifiable. The approximate solutions obtained upon factorization may be further improved by the following method. Let S_o be an approximate solution of the characteristic equation, and assume that $(1+\alpha)S_o$, where α is a small (real or complex) number, is the exact solution. Substitute $(1+\alpha)S_o$ in the characteristic equation, then use the approximation $(1+\alpha)^n \approx 1+n\alpha$ to linearize the resulting equation. You will find an expression for α, which corresponds to a more accurate solution than S_o, provided, of course, that the magnitude of the computed α is well below unity.