Problem 12) a) Particle m_1: $m_1 x_1''(t) = q_1 E_{xo}\cos(\omega t) - \alpha_1[x_1(t) - x_0(t)] - \beta_1 x_1'(t)$, (1a)

Particle m_2: $m_2 x_2''(t) = q_2 E_{xo}\cos(\omega t) - \alpha_2[x_2(t) - x_0(t)] - \beta_2 x_2'(t)$, (1b)

Particle M: $M x_0''(t) = -(q_1+q_2) E_{xo}\cos(\omega t) + \alpha_1[x_1(t) - x_0(t)] + \alpha_2[x_2(t) - x_0(t)] + \beta_1 x_1'(t) + \beta_2 x_2'(t)$. (1c)

Adding Eq.(1a) to Eq.(1b), then using the fact of stationarity of the center-of-mass, namely, $m_1 x_1(t) + m_2 x_2(t) + M x_0(t) = 0$, leads to Eq.(1c), which is thereby revealed as containing no new information.

b) Particle m_1: $-m_1\omega^2 x_{10} + \alpha_1(x_{10} - x_{00}) - \mathrm{i}\omega\beta_1 x_{10} = q_1 E_{xo}$, (2a)

Particle m_2: $-m_2\omega^2 x_{20} + \alpha_2(x_{20} - x_{00}) - \mathrm{i}\omega\beta_2 x_{20} = q_2 E_{xo}$. (2b)

c) Particle m_1: $\omega^2 x_{10} - (\alpha_1/m_1)[x_{10} + (m_1/M)x_{10} + (m_2/M)x_{20}] + \mathrm{i}\omega(\beta_1/m_1)x_{10} = -(q_1/m_1)E_{xo}$, (3a)

Particle m_2: $\omega^2 x_{20} - (\alpha_2/m_2)[x_{20} + (m_1/M)x_{10} + (m_2/M)x_{20}] + \mathrm{i}\omega(\beta_2/m_2)x_{20} = -(q_2/m_2)E_{xo}$. (3b)

d) The above equations may be written in matrix form, as follows :

$$\begin{bmatrix} \omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1) - (\alpha_1/M) & -(\alpha_1 m_2/M m_1) \\ -(\alpha_2 m_1/M m_2) & \omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2) - (\alpha_2/M) \end{bmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = -\begin{pmatrix} q_1/m_1 \\ q_2/m_2 \end{pmatrix} E_{xo}. \qquad (4)$$

Inverting the 2×2 coefficient matrix in Eq.(4) yields the following solutions for x_{10} and x_{20}:

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = -\{[\omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1)][\omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2)]$$
$$-(\alpha_1/M)[\omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2)] - (\alpha_2/M)[\omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1)]\}^{-1}$$
$$\times \begin{bmatrix} \omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2) - (\alpha_2/M) & (\alpha_1 m_2/M m_1) \\ (\alpha_2 m_1/M m_2) & \omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1) - (\alpha_1/M) \end{bmatrix} \begin{pmatrix} q_1/m_1 \\ q_2/m_2 \end{pmatrix} E_{xo} \qquad (5)$$

e) We have $p_o = q_1 x_{10} + q_2 x_{20} - (q_1+q_2)x_{00} = [q_1 + (q_1+q_2)(m_1/M)]x_{10} + [q_2 + (q_1+q_2)(m_2/M)]x_{20}$. Substituting for x_{10} and x_{20} from Eq.(5) yields

$$p_o/E_{xo} = -\{[q_1^2(1/m_1 + 1/M) + q_1 q_2/M][\omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2)] + [q_2^2(1/m_2 + 1/M) + q_1 q_2/M]$$
$$\times[\omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1)] - (\alpha_1 q_2 - \alpha_2 q_1)(m_1 q_2 - m_2 q_1)/(M m_1 m_2)\}$$
$$/\{[\omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1)][\omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2)]$$
$$-(\alpha_1/M)[\omega^2 + \mathrm{i}\omega(\beta_2/m_2) - (\alpha_2/m_2)] - (\alpha_2/M)[\omega^2 + \mathrm{i}\omega(\beta_1/m_1) - (\alpha_1/m_1)]\}. \qquad (6)$$

f) In the special case when $\alpha_1/m_1 = \alpha_2/m_2 = \omega_o^2$ and $\beta_1/m_1 = \beta_2/m_2 = \gamma$, Eq.(6) simplifies as follows:

$$p_o/E_{xo} = -\frac{[q_1^2/m_1 + q_2^2/m_2 + (q_1+q_2)^2/M](\omega^2 + \mathrm{i}\gamma\omega - \omega_o^2) - (m_2 q_1^2/m_1 + m_1 q_2^2/m_2 - 2q_1 q_2)(\omega_o^2/M)}{(\omega^2 + \mathrm{i}\gamma\omega - \omega_o^2)[\omega^2 + \mathrm{i}\gamma\omega - (m_1 + m_2 + M)(\omega_o^2/M)]}. \qquad (7a)$$

Further manipulation of Eq. (7a) then yields

$$p_o/E_{xo} = -\frac{(q_1^2/m_1)+(q_2^2/m_2)-(q_1+q_2)^2/(m_1+m_2)}{\omega^2+\mathrm{i}\gamma\omega-\omega_o^2} - \frac{(q_1+q_2)^2[1/(m_1+m_2)+(1/M)]}{\omega^2+\mathrm{i}\gamma\omega-\omega_o^2[1+(m_1+m_2)/M]}. \qquad (7b)$$

It is clear from Eq. (7b) that, in the case under consideration, the coupled oscillator is the superposition of two simple oscillators with resonance frequencies ω_o and $\omega_o\sqrt{1+(m_1+m_2)/M}$, both having the same damping coefficient γ. Note also that, in the limit when $M\to\infty$, the coupling between the two dipoles vanishes, reducing Eq. (7b) to the sum of two single-dipole oscillators with the same resonance frequency ω_o, the same damping coefficient γ, and with respective numerators q_1^2/m_1 and q_2^2/m_2, as expected.

Finally, compare the resonance frequencies of two coupled identical oscillators, namely, ω_o and $\omega_o\sqrt{1+(2m/M)}$ derived in the preceding paragraphs, with the resonance frequency $\omega_o\sqrt{1+(m/M)}$ of each individual oscillator when uncoupled (see Problem 6). Clearly, the coupling splits the resonance frequency, pushing one frequency lower and the other one higher.