Problem 9)

a) $$C(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{\omega_p^2(\omega_0^2 - \omega^2) + i\omega_p^2\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}.$$

The denominator is always positive, and $\omega_p^2\gamma\omega \geq 0. \Rightarrow \operatorname{Im}\{C(\omega)\} \geq 0$.
In the numerator, the real part is $\omega_p^2(\omega_0^2 - \omega^2)$, which is positive if $\omega_0 > \omega$, zero if $\omega_0 = \omega$, and negative if $\omega_0 < \omega$. Therefore, $\operatorname{Re}\{C(\omega)\}$ could be positive, zero, or negative.

b) $$\chi(\omega) = \frac{3C(\omega)}{3 - C(\omega)} = 3\,\frac{c' + ic''}{3 - c' - ic''} = 3\,\frac{(c'+ic'')(3-c'+ic'')}{(3-c')^2 + c''^2} =$$

$$3\,\frac{(3-c')c' - c''^2 + ic'c'' + ic''(3-c')}{(3-c')^2 + c''^2} = 3\,\frac{3c' - (c'^2 + c''^2) + i3c''}{(3-c')^2 + c''^2} \Rightarrow$$

$$\operatorname{Im}\{\chi(\omega)\} = \frac{9c''}{(3-c')^2 + c''^2} \geq 0$$ (because denominator is positive and c'' in the numerator is ≥ 0.)

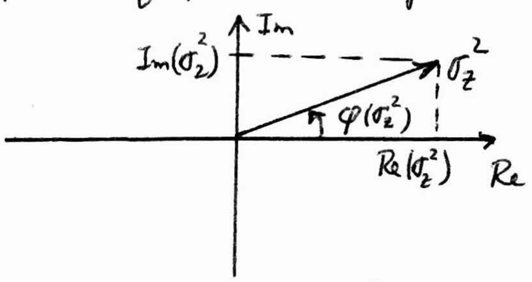
As for the real part of $\chi(\omega)$, the sign is determined by $3c' - (c'^2 + c''^2)$. This will be negative when $c' < 0$, and can be zero or positive for many combinations of $\omega, \omega_0, \omega_p, \gamma$. Therefore, $\operatorname{Re}\{\chi(\omega)\}$ can be positive, zero, or negative.

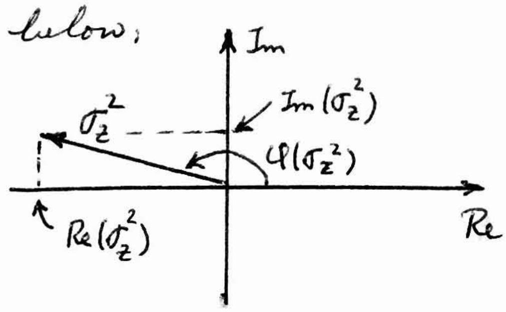
c) $\sigma_z^2 = 1 + \chi(\omega) - \sigma_x^2 - \sigma_y^2$ has real and imaginary parts given by

$$\operatorname{Re}\{\sigma_z^2\} = \operatorname{Re}\{\chi(\omega)\} + (1 - \sigma_x^2 - \sigma_y^2); \qquad \operatorname{Im}\{\sigma_z^2\} = \operatorname{Im}\{\chi(\omega)\} \geq 0.$$

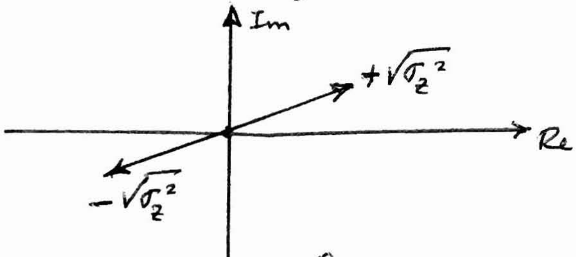
Clearly, the real part of σ_z^2 can be positive, zero, or negative, whereas the

Imaginary parts of σ_z^2 must always be non-negative. Possible locations of σ_z^2 in the complex-plane are, therefore, always in the upper half-plane. Examples are shown below:





The polar angle of σ_z^2, denoted by $\varphi(\sigma_z^2)$ in the above diagrams, is between zero and $180°$. This means that the polar angle of $\sigma_z = \sqrt{\sigma_z^2}$ is between $0°$ and $90°$. Therefore, $\sigma_z = +\sqrt{\sigma_z^2}$ is in the first quadrant, while $\sigma_z = -\sqrt{\sigma_z^2}$ is in the third quadrant, as shown below:



Clearly, one possible value of σ_z has $Re(\sigma_z) \geq 0$ and $Im(\sigma_z) \geq 0$, whereas the other possible value has $Re(\sigma_z) \leq 0$ and $Im(\sigma_z) \leq 0$. In general <u>both</u> values of σ_z are acceptable, unless the plane-wave happens to be in a semi-infinite medium where either $z \to \infty$ or $z \to -\infty$. Now the exponential function of the plane-wave is:

$$\exp[i(k_0 \vec{\sigma} \cdot \vec{r} - \omega t)] = \exp[-k_0 Im(\sigma_z) z] \exp[ik_0(\sigma_x x + \sigma_y y + Re(\sigma_z) z - ct)].$$

Therefore, if $z \to +\infty$, the acceptable solution must have $Im(\sigma_z) \geq 0$; <u>in this case the "+ solution" is acceptable</u>. On the other hand, <u>if $z \to -\infty$, the acceptable solution must have $Im(\sigma_z) \leq 0$; that is, the "− solution" will be acceptable</u>. If the medium in which the plane-wave resides happens to have a finite thickness along the z-axis, then <u>both the "+ solution" and the "− solution" will coexist</u> within the medium.