Problem 6.8)

a) Energy density $= N\int_{E=0}^{E_0} \mathrm{d}\mathcal{E} = N\int_{E=0}^{E_0} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{p} = \int_{E=0}^{E_0} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{P} = \int_{E=0}^{E_0} \boldsymbol{E}\cdot \mathrm{d}[\varepsilon_o\,\chi(0)\boldsymbol{E}]$

$$= \varepsilon_o\,\chi(0)\int_{E=0}^{E_0} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{E} = \tfrac{1}{2}\varepsilon_o\,\chi(0)E^2\Big|_0^{E_0} = \tfrac{1}{2}\varepsilon_o\,\chi(0)E_0^2. \tag{1}$$

In the above equation, N is the number of dipoles per unit volume, and, as usual, $\boldsymbol{P} = N\boldsymbol{p}$. Note that $\chi(0) = Nq^2/(m\varepsilon_o\omega_0^2) = Nq^2/(\varepsilon_o\alpha)$, where α is the spring constant. Consequently,

$$\text{Energy density} = \tfrac{1}{2}\,N(qE_0)^2/\alpha. \tag{2}$$

In the steady state, $qE_0 = \alpha d$, where d is the length of the dipole. In other words, the force of the E-field acting on the negative charge, $-qE_0$, is balanced by the force of the spring (spring constant $= \alpha$) exerted on the negative charge when the length of the spring is d. The energy density stored within the springs is thus given by $\tfrac{1}{2}N\alpha d^2$. This is readily recognized as the potential energy of N springs, each having a constant α and stretched to length d.

b) Total energy density = E-field's energy density + Dipoles' energy density

$$= \tfrac{1}{2}\varepsilon_o E_0^2 + \tfrac{1}{2}\varepsilon_o\chi(0)E_0^2 = \tfrac{1}{2}\varepsilon_o[1+\chi(0)]E_0^2 = \tfrac{1}{2}\varepsilon_o\varepsilon(0)E_0^2. \tag{3}$$

c) Energy density per unit volume of dipoles $= \int_{E=0}^{E_0\cos(\omega t+\varphi_0)} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{P}$

$$= \varepsilon_o\,\chi(\omega)\int_{E=0}^{E_0\cos(\omega t+\varphi_0)} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{E}$$

$$= \tfrac{1}{2}\varepsilon_o\,\chi(\omega)E_0^2\cos^2(\omega t+\varphi_0). \tag{4}$$

The above expression yields the time-dependent energy-density of the dipoles. The dipoles gain internal energy when elongated under the influence of the E-field. When the E-field returns to zero, the dipoles shrink, returning their internal energy to the system in the form of radiation.

$$\text{Total energy density} = \tfrac{1}{2}\varepsilon_o E_0^2\cos^2(\omega t+\varphi_0) + \tfrac{1}{2}\varepsilon_o\chi(\omega)E_0^2\cos^2(\omega t+\varphi_0)$$

$$= \tfrac{1}{2}\varepsilon_o\varepsilon(\omega)E_0^2\cos^2(\omega t+\varphi_0).$$

$$\Longrightarrow \quad \text{Time averaged energy density} = \tfrac{1}{4}\varepsilon_o\varepsilon(\omega)|E_0|^2. \tag{5}$$

Digression: In response to the E-field $\boldsymbol{E}(t) = E_0\hat{\boldsymbol{x}}\cos(\omega t+\varphi_0)$, a single dipole will oscillate with frequency ω, the distance between its $\pm q$ charges being $x(t) = x_0\cos(\omega t+\varphi_0)$. The oscillation amplitude x_0 is given by Eq.(2b) of Chapter 6 ($\gamma = 0$ in the absence of dissipative losses). The oscillating dipole's mechanical energy will then be given by the sum of its kinetic and potential energies, as follows:

$$\mathcal{E}_{\text{dipole}}(t) = \mathcal{E}_k(t) + \mathcal{E}_p(t) = \tfrac{1}{2}mv^2(t) + \tfrac{1}{2}\alpha x^2(t) = \tfrac{1}{2}mx_0^2\omega^2\sin^2(\omega t+\varphi_0) + \tfrac{1}{2}\alpha x_0^2\cos^2(\omega t+\varphi_0)$$

$$= \tfrac{1}{2}mx_0^2\omega^2 + \tfrac{1}{2}m(\omega_0^2-\omega^2)x_0^2\cos^2(\omega t+\varphi_0)$$

$$= \tfrac{1}{2}mx_0^2\omega^2 + \tfrac{1}{2}\left(\frac{q^2/m}{\omega_0^2-\omega^2}\right)E_{x0}^2\cos^2(\omega t+\varphi_0). \tag{6}$$

Multiplying both sides of Eq.(6) by the number-density N of the dipoles, we find the total mechanical energy-density associated with the dipoles to be

$$\mathcal{E}(t) = ½Nmx_0^2\omega^2 + ½\varepsilon_o\chi(\omega)E_{x0}^2\cos^2(\omega t + \varphi_0). \tag{7}$$

The first term on the right-hand side of the above equation is a (time-independent) background energy representing the mechanical energy-density imparted to the dipoles when the exciting field is initially established within the host medium; this term has been ignored in our derivation of the energy-density of the dipoles in part (c). The second term appearing on the right-hand side of Eq.(7) represents the continually exchanged energy-density between the E-field and the dipoles. This latter term, of course, is the same expression that was derived in Eq.(4).

---

d) $$\text{Dipoles' energy density} = \int_{E=0}^{E(t)} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{P} = \int_{t'=0}^{t} \boldsymbol{E}(t')\cdot\frac{\mathrm{d}\boldsymbol{P}(t')}{\mathrm{d}t'}\mathrm{d}t'$$

$$= \varepsilon_o E_0^2\int_0^t[\sin(\omega_1 t') - \sin(\omega_2 t')][\omega_1\chi(\omega_1)\cos(\omega_1 t') - \omega_2\chi(\omega_2)\cos(\omega_2 t')]\mathrm{d}t'$$

$$= \varepsilon_o E_0^2\int_0^t[\omega_1\chi_1\sin(\omega_1 t')\cos(\omega_1 t') + \omega_2\chi_2\sin(\omega_2 t')\cos(\omega_2 t')$$
$$-\omega_1\chi_1\sin(\omega_2 t')\cos(\omega_1 t') - \omega_2\chi_2\sin(\omega_1 t')\cos(\omega_2 t')]\mathrm{d}t'$$

$$= ½\varepsilon_o E_0^2\Big\{\int_0^t[\omega_1\chi_1\sin(2\omega_1 t') + \omega_2\chi_2\sin(2\omega_2 t')\,]\mathrm{d}t'$$
$$-\int_0^t\omega_1\chi_1\{\sin[(\omega_1+\omega_2)t'] - \sin[(\omega_1-\omega_2)t']\}\mathrm{d}t'$$
$$-\int_0^t\omega_2\chi_2\{\sin[(\omega_1+\omega_2)t'] + \sin[(\omega_1-\omega_2)t']\}\mathrm{d}t'\Big\}$$

$$= ½\varepsilon_o E_0^2\{[-½\chi_1\cos(2\omega_1 t') - ½\chi_2\cos(2\omega_2 t')]_0^t$$
$$+[(\omega_1\chi_1+\omega_2\chi_2)/(\omega_1+\omega_2)]\cos[(\omega_1+\omega_2)t']_0^t$$
$$-[(\omega_2\chi_2-\omega_1\chi_1)/(\omega_2-\omega_1)]\cos[(\omega_1-\omega_2)t']_0^t\}$$

$$= ½\varepsilon_o E_0^2\{-½\chi_1\cos(2\omega_1 t) + ½\chi_1 - ½\chi_2\cos(2\omega_2 t) + ½\chi_2$$
$$+[(\omega_1\chi_1+\omega_2\chi_2)/(\omega_1+\omega_2)]\{\cos[(\omega_1+\omega_2)t] - 1\}$$
$$-[(\omega_2\chi_2-\omega_1\chi_1)/(\omega_2-\omega_1)]\{\cos[(\omega_1-\omega_2)t] - 1\}\}. \tag{8}$$

Note that $\cos(2\omega_1 t) = \cos[(2m-1)(\Delta\omega)t]$ has an integer number of periods between $t = 0$ and $t = T = 2\pi/\Delta\omega$; therefore, time-averaging over the interval $[0, T]$ eliminates the term containing $\cos(2\omega_1 t)$. Similarly, $\cos(2\omega_2 t) = \cos[(2m+1)(\Delta\omega)t]$ vanishes upon averaging. The same is true of $\cos[(\omega_1+\omega_2)t] = \cos[2m(\Delta\omega)t]$ and $\cos[(\omega_1-\omega_2)t] = \cos[(\Delta\omega)t]$. Consequently,

Time-averaged energy density of dipoles over the interval $[0, T]$

$$= ½\varepsilon_o E_0^2\{½\chi_1 + ½\chi_2 - [(\omega_1\chi_1+\omega_2\chi_2)/(\omega_1+\omega_2)] + [(\omega_2\chi_2-\omega_1\chi_1)/(\omega_2-\omega_1)]\}$$

$$= ½\varepsilon_o E_0^2\left[\frac{\chi(\omega_1)+\chi(\omega_2)}{2} - \frac{(m-½)\Delta\omega\chi(\omega_1) + (m+½)\Delta\omega\chi(\omega_2)}{2m\Delta\omega} + \frac{(m+½)\Delta\omega\chi(\omega_2) - (m-½)\Delta\omega\chi(\omega_1)}{\Delta\omega}\right]$$

$$= ½\varepsilon_o E_0^2\left[\frac{\chi(\omega_1)-\chi(\omega_2)}{4m} + \frac{\chi(\omega_1)+\chi(\omega_2)}{2} + (m\Delta\omega)\frac{\chi(\omega_2)-\chi(\omega_1)}{\Delta\omega}\right]. \tag{9}$$

In the limit when $\Delta\omega \to 0$, we will have $m \gg 1$, in which case the first term on the right-hand side of Eq.(9) may be ignored. Denoting by ω_c the central frequency $½(\omega_1 + \omega_2) = m\Delta\omega$, we now write the time-averaged energy-density of the dipoles as follows:

$$\langle\text{Dipoles' energy density}\rangle \cong ½\varepsilon_o E_0^2 \left[\chi(\omega_c) + \omega_c \frac{\mathrm{d}\chi(\omega)}{\mathrm{d}\omega}\bigg|_{\omega=\omega_c}\right]. \tag{10}$$

Next, we compute the time-averaged E-field energy density over the interval $[0, T]$, that is,

$$\begin{aligned}\langle ½\varepsilon_o E^2(t)\rangle &= \frac{\varepsilon_o E_0^2}{2T}\int_0^T [\sin(\omega_1 t) - \sin(\omega_2 t)]^2 \mathrm{d}t \\ &= \frac{\varepsilon_o E_0^2}{2T}\int_0^T [\sin^2(\omega_1 t) + \sin^2(\omega_2 t) - 2\sin(\omega_1 t)\sin(\omega_2 t)]\mathrm{d}t \\ &= \frac{\varepsilon_o E_0^2}{2T}\int_0^T \{½[1 - \cos(2\omega_1 t)] + ½[1 - \cos(2\omega_2 t)] \\ &\qquad\qquad + \cos[(\omega_1 + \omega_2)t] - \cos[(\omega_1 - \omega_2)t]\}\mathrm{d}t \\ &= ½\varepsilon_o E_0^2. \end{aligned} \tag{11}$$

We thus find: $\langle\text{Total energy density}\rangle = \langle E \text{ field energy density}\rangle + \langle\text{Dipoles' energy density}\rangle$

$$\begin{aligned} &= ½\varepsilon_o E_0^2[1 + \chi(\omega_c) + \omega_c \chi'(\omega_c)] \\ &= ½\varepsilon_o[\varepsilon(\omega_c) + \omega_c \varepsilon'(\omega_c)]E_0^2 \\ &= ½\varepsilon_o \left(\frac{d[\omega\varepsilon(\omega)]}{\mathrm{d}\omega}\bigg|_{\omega_c}\right) E_0^2. \end{aligned} \tag{12}$$

Noting that the time-averaged E-field intensity over the beat period T is $\langle E^2(t)\rangle = E_0^2$, we have

$$\langle\text{Total energy density associated with } E \text{ field}\rangle = ½\varepsilon_o \left(\frac{d[\omega\varepsilon(\omega)]}{\mathrm{d}\omega}\bigg|_{\omega_c}\right)\langle E^2(t)\rangle. \tag{13}$$

Digression: As a check on the above result, consider a *quasi-monochromatic* plane-wave having E-field amplitude $\tilde{E}_0\hat{\boldsymbol{x}}$, H-field amplitude $\tilde{H}_0\hat{\boldsymbol{y}} = n(\omega)\tilde{E}_0\hat{\boldsymbol{y}}/Z_0$, propagating in a transparent, dispersive, non-magnetic medium of refractive index $n(\omega) = \sqrt{\varepsilon(\omega)} = \sqrt{1 + \chi(\omega)}$. We will have

$$\begin{aligned}\langle E \text{ field energy density}\rangle + \langle H \text{ field energy density}\rangle &= ¼\varepsilon_o[\varepsilon(\omega) + \omega\varepsilon'(\omega)]\tilde{E}_0^2 + ¼\mu_o\tilde{H}_0^2 \\ &= ¼\varepsilon_o[\varepsilon(\omega) + \omega\varepsilon'(\omega) + n^2(\omega)]\tilde{E}_0^2 = ½\varepsilon_o n(\omega)\left[n(\omega) + \frac{\omega\varepsilon'(\omega)}{2n(\omega)}\right]\tilde{E}_0^2 \\ &= \frac{n(\omega)}{2Z_0 c}[n(\omega) + \omega n'(\omega)]\tilde{E}_0^2 = \langle S_z\rangle/V_g. \end{aligned} \tag{14}$$

In words, the average EM energy-density, when multiplied by the group velocity $V_g = c/n_g(\omega) = c/[n(\omega) + \omega n'(\omega)] = c/[\omega n(\omega)]'$ yields the time-averaged component $\langle S_z\rangle$ of the Poynting vector along the direction of propagation, as it should.
