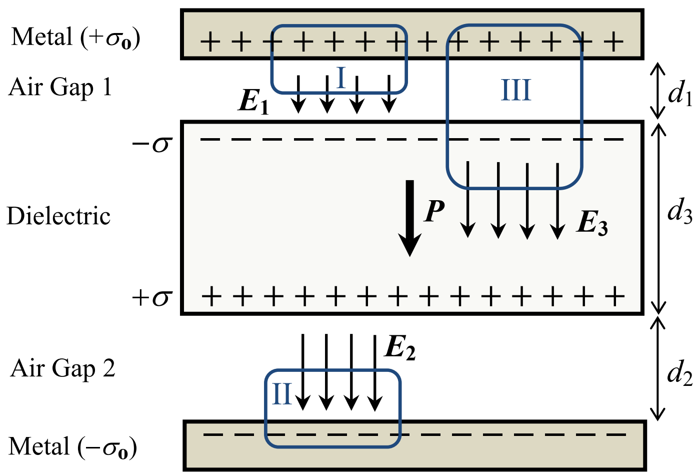
Problem 6.7)

a) Invoking Gauss's law, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho/\varepsilon_o$, together with the fact that the E-field inside a metallic plate is zero, pillboxes I and II enable us to determine $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ as $\boldsymbol{E}_1=\boldsymbol{E}_2=-(\sigma_0/\varepsilon_o)\hat{\boldsymbol{z}}$.



b) The bound electric charge-densities at the upper and lower facets of the dielectric slab are obtained from the relation $\rho_{\text{bound}}^{(e)}=-\boldsymbol{\nabla}\cdot\boldsymbol{P}$ as $\sigma=P$. (Both σ and P have units of coulomb/m^2.) The linearity of $\boldsymbol{P}$ in relation to the E-field now allows one to write $\sigma=P=\varepsilon_o\chi E_3$.

c) Use pillbox III in conjunction with Gauss's law and the fact that the E-field inside the metallic conductor is zero to obtain $\boldsymbol{E}_3=-(\sigma_0-\sigma)\hat{\boldsymbol{z}}/\varepsilon_o$. Next substitute for σ in terms of E_3 to find

$$E_3=(\sigma_0-\varepsilon_o\chi E_3)/\varepsilon_o \quad\rightarrow\quad (1+\chi)E_3=\sigma_0/\varepsilon_o \quad\rightarrow\quad \boldsymbol{E}_3=-\sigma_0\hat{\boldsymbol{z}}/[\varepsilon_o(1+\chi)].$$

d) Inside the dielectric slab, $\boldsymbol{D}_3=\varepsilon_o\boldsymbol{E}_3+\boldsymbol{P}=\varepsilon_o\boldsymbol{E}_3+\varepsilon_o\chi\boldsymbol{E}_3=\varepsilon_o(1+\chi)\boldsymbol{E}_3$. With reference to pillbox III, since the D-field inside the metallic plate is zero, and since the free charges contained within the pillbox have a surface charge-density σ_0, Gauss's law $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$ implies that $\boldsymbol{D}_3=-\sigma_0\hat{\boldsymbol{z}}$, which yields the same expression for $\boldsymbol{E}_3$ as that obtained in part (c).

In the free-space regions of the gap, $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}$ (because $\boldsymbol{P}=0$), and the application of Gauss's law to pillboxes I and II yields $\boldsymbol{D}_1=\boldsymbol{D}_2=-\sigma_0\hat{\boldsymbol{z}}$, yielding $\boldsymbol{E}_1=\boldsymbol{E}_2=-(\sigma_0/\varepsilon_o)\hat{\boldsymbol{z}}$, as was found in part (a).

e) $$\text{Voltage}=V_{\text{top}}-V_{\text{bottom}}=\int_{\text{top}}^{\text{bottom}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{\ell}=E_1d_1+E_2d_2+E_3d_3$$
$$=(\sigma_0/\varepsilon_o)(d_1+d_2)+\sigma_0 d_3/[\varepsilon_o(1+\chi)]$$
$$=(\sigma_0/\varepsilon_o)\left(d_1+d_2+\frac{d_3}{1+\chi}\right).$$

free charges only

f) $$C=Q/V=\frac{\sigma_0 A}{(\sigma_0/\varepsilon_o)[d_1+d_2+d_3/(1+\chi)]}=\frac{\varepsilon_o A}{d_1+d_2+[d_3/(1+\chi)]}.$$

In the absence of the dielectric slab, $\chi = 0$ and $C = \varepsilon_0 A/(d_1 + d_2 + d_3)$. Considering that $\chi > 0$, the presence of the slab reduces the effective gap (by dividing d_3 by $1 + \chi$), thus *increasing* the capacitance C.

g) In general, the electrical power delivered to the capacitor is $\wp(t) = V(t)I(t)$. Therefore,

$$\text{Delivered Energy} = \int_0^\infty V(t)I(t)\mathrm{d}t = C\int_0^\infty V(t)\mathrm{d}V(t) = ½CV^2(t)\Big|_{t=0}^{\infty} = ½CV^2(\infty)$$

$$= ½C(Q/C)^2 = Q^2/(2C).$$

h) Using the expression for the capacitance C found in part (f), we write

$$\text{Delivered Energy} = \text{Stored Energy} = \frac{Q^2}{2\varepsilon_0 A}\left(d_1 + d_2 + \frac{d_3}{1+\chi}\right)$$

$$= ½\varepsilon_0 A\left(\frac{Q}{\varepsilon_0 A}\right)^2\left(d_1 + d_2 + \frac{d_3}{1+\chi}\right)$$

$$= ½\varepsilon_0 A d_1(\sigma_0/\varepsilon_0)^2 + ½\varepsilon_0 A d_2(\sigma_0/\varepsilon_0)^2 + ½\varepsilon_0 A d_3(1+\chi)\left[\frac{\sigma_0}{\varepsilon_0(1+\chi)}\right]^2$$

$$= ½\varepsilon_0|\boldsymbol{E}_1|^2 A d_1 + ½\varepsilon_0|\boldsymbol{E}_2|^2 A d_2 + ½\varepsilon_0(1+\chi)|\boldsymbol{E}_3|^2 A d_3.$$

Now, Ad_1, Ad_2, and Ad_3 are the volumes of the three regions of space between the capacitor plates. Within the air-gaps, the energy-densities are $½\varepsilon_0|\boldsymbol{E}_1|^2$ and $½\varepsilon_0|\boldsymbol{E}_2|^2$. The remaining energy is contained in the dielectric slab, and is proportional to its volume Ad_3. Consequently, the density of the E-field energy within the slab must be $½\varepsilon_0(1+\chi)|\boldsymbol{E}_3|^2$.

Note: The above expression for the E-field energy-density within a dielectric medium is valid only for dispersionless media, that is, media for which the frequency-dependence of $\chi(\omega)$ can be ignored. When the local variations of $\chi(\omega)$ with ω are significant (i.e., $\mathrm{d}\chi(\omega)/\mathrm{d}\omega \neq 0$), the energy-density expression must be modified to account for the medium's dispersive properties.