Problem 5) $\vec{E}(\vec{r},t) = \text{Real}\{\vec{E}(\vec{r},\omega)e^{-i\omega t}\} = \vec{E}'(\vec{r},\omega)\cos\omega t + \vec{E}''(\vec{r},\omega)\sin\omega t.$

Note that $\vec{E}'(\vec{r},\omega)$ and $\vec{E}''(\vec{r},\omega)$ are real-valued vectors in 3D-space.

$$\vec{P}(\vec{r},t) = \text{Real}\{\vec{P}(\vec{r},\omega)e^{-i\omega t}\} = \text{Real}\{\chi(\omega)\vec{E}(\vec{r},\omega)e^{-i\omega t}\}$$

$$= \text{Real}\{(\chi'+i\chi'')(\vec{E}'+i\vec{E}'')(\cos\omega t - i\sin\omega t)\}$$

$$= \text{Real}\{[(\chi'\vec{E}'-\chi''\vec{E}'')+i(\chi'\vec{E}''+\chi''\vec{E}')](\cos\omega t - i\sin\omega t)\} \Rightarrow$$

$$\vec{P}(\vec{r},t) = (\chi'\vec{E}'-\chi''\vec{E}'')\cos\omega t + (\chi'\vec{E}''+\chi''\vec{E}')\sin\omega t.$$

a) $\dfrac{\partial \mathcal{E}(\vec{r},t)}{\partial t} = \vec{E}(\vec{r},t)\cdot\dfrac{\partial}{\partial t}\vec{P}(\vec{r},t) = E^T(\vec{r},t)\dfrac{\partial}{\partial t}P(\vec{r},t) \leftarrow$ Linear algebraic notation; $\vec{E}$ and $\vec{P}$ are written as column vectors; T is for transpose.

$$= (E'^T\cos\omega t + E''^T\sin\omega t)\left[-\omega(\chi'E'-\chi''E'')\sin\omega t + \omega(\chi'E''+\chi''E')\cos\omega t\right]$$

$$= \omega(-E'^T\chi'E' + E'^T\chi''E'' + E''^T\chi'E'' + E''^T\chi''E')\sin\omega t\cos\omega t - \omega(E''^T\chi'E' - E''^T\chi''E'')\sin^2\omega t$$

$$+ \omega(E'^T\chi'E'' + E'^T\chi''E')\cos^2\omega t$$

$\cos^2\omega t \to \dfrac{1+\cos 2\omega t}{2}$, $\quad \sin^2\omega t \to \dfrac{1-\cos 2\omega t}{2}$

$$= \tfrac{1}{2}\omega(-E'^T\chi'E' + E''^T\chi'E'' + E'^T\chi''E'' + E''^T\chi''E')\sin(2\omega t)$$

$$+ \tfrac{1}{2}\omega(E'^T\chi''E' - E''^T\chi''E'' + E'^T\chi'E'' + E''^T\chi'E')\cos(2\omega t)$$

$$+ \tfrac{1}{2}\omega(E'^T\chi''E' + E''^T\chi''E'' + E'^T\chi'E'' - E''^T\chi'E')$$

b) Period-averaging $\partial\mathcal{E}(\vec{r},t)/\partial t$ results in $\langle\sin(2\omega t)\rangle = \langle\cos(2\omega t)\rangle = 0$. We also use the fact that $E''^T\chi'E' = E'^T\chi'^T E''$ to write:

$$\langle\partial\mathcal{E}(\vec{r},t)/\partial t\rangle = \tfrac{1}{2}\omega\left[E'^T\chi''E' + E''^T\chi''E'' + E'^T(\chi'-\chi'^T)E''\right]$$

First suppose $\chi''=0$ at some frequency ω. The remaining term $E'^T(\chi'-\chi'^T)E''$ must be ≥ 0 for all choices of E' and E''. Changing the sign of either E' or E'' will then change the sign of $E'(\chi'-\chi'^T)E''$; but this is impossible if the energy transfer to the local polarization density is supposed to be positive. We conclude that $E'^T(\chi'-\chi'^T)E''=0$ for any choice of $\vec{E}'$ and $\vec{E}''$. Consequently $\chi'-\chi'^T=0$, i.e., χ' is a symmetric matrix.

Similarly, even when $\chi''(\omega)\neq 0$, if there is a chance that a sign-change of $\vec{E}'$ or $\vec{E}''$ will make $\langle\partial\mathcal{E}/\partial t\rangle$ negative, χ' must be symmetric. Changing the sign of $\vec{E}'$ or $\vec{E}''$ causes an elliptically-polarized state to switch from the right to the left, or vice-versa. If absorption in the material happens to be the same for both right- and left-polarized states, we must have a symmetric $\chi'(\omega)$.

Next assume that either $\vec{E}'=0$ or $\vec{E}''=0$. The last term in the expression for $\langle\partial\mathcal{E}/\partial t\rangle$ will then disappear, and we must have $E'^T\chi''E'\geq 0$ or $E''^T\chi''E''\geq 0$. Since $\vec{E}'$ and $\vec{E}''$ are arbitrary, we conclude that χ'' must be a positive-semidefinite matrix, namely, that for any real-valued vector $\vec{E}\neq 0$, we must have $E^T\chi''E\geq 0$.

Any matrix such as χ'' can be decomposed into symmetric and anti-symmetric parts, namely, $\chi''=\chi''_s+\chi''_a$, where $\chi''_s=\frac{1}{2}(\chi''+\chi''^T)$ and $\chi''_a=\frac{1}{2}(\chi''-\chi''^T)$. Note that $\chi''^T_s=\chi''_s$ whereas $\chi''^T_a=-\chi''_a$. We may write

$$E^T\chi''E=E^T\chi''_sE+E^T\chi''_aE.$$

But $(E^T\chi''_aE)^T=E^T\chi''^T_aE=-E^T\chi''_aE$. Therefore $E^T\chi''_aE=0$. Thus any anti-symmetric part of $\chi''(\omega)$ <u>cannot</u> contribute to absorption. When $\chi'(\omega)$ is symmetric and $\chi''(\omega)$ is anti-symmetric, $\chi(\omega)=\chi'(\omega)+i\chi''(\omega)$ will be Hermitian, i.e., $\chi^T(\omega)=\chi^*(\omega)$. Under these circumstances, the material will be transparent (i.e., non-absorbing). Any absorption must then result from the symmetric part of $\chi''(\omega)$, which is necessarily positive-definite.