5-54) Upon introducing $\rho_{\text{total}}^{(e)}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t) - \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$ as the total electric charge-density, and $\boldsymbol{J}_{\text{total}}^{(e)}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t + \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)$ as the total electric current-density, Maxwell's equations are streamlined as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \varepsilon_0^{-1}\rho_{\text{total}}^{(e)}, \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}_{\text{total}}^{(e)} + \mu_0\varepsilon_0\partial\boldsymbol{E}/\partial t, \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{4}$$

To eliminate the B-field, we apply the curl operator to both sides of Eq.(3), and arrive at

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = -\partial(\boldsymbol{\nabla}\times\boldsymbol{B})/\partial t. \tag{5}$$

Substitution for $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})$ in terms of the Laplacian, and also for $\boldsymbol{\nabla}\times\boldsymbol{B}$ from Maxwell's 2nd equation, yields

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \boldsymbol{\nabla}^2\boldsymbol{E} = -\mu_0\frac{\partial\boldsymbol{J}_{\text{total}}^{(e)}}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}. \tag{6}$$

Next, we substitute for $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ from Maxwell's 1st equation, then rearrange the terms to arrive at

$$\boldsymbol{\nabla}^2\boldsymbol{E} - \frac{\partial^2\boldsymbol{E}}{c^2\partial t^2} = \varepsilon_0^{-1}\boldsymbol{\nabla}\rho_{\text{total}}^{(e)} + \mu_0\frac{\partial\boldsymbol{J}_{\text{total}}^{(e)}}{\partial t}. \tag{7}$$

In a similar vein, we search for a 2nd order partial differential equation for $\boldsymbol{B}(\boldsymbol{r},t)$ that relates the B-field to its source $\boldsymbol{J}_{\text{total}}^{(e)}$. Applying the curl operator to both sides of Eq.(2), we find

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mu_0\boldsymbol{\nabla}\times\boldsymbol{J}_{\text{total}}^{(e)} + \mu_0\varepsilon_0\partial(\boldsymbol{\nabla}\times\boldsymbol{E})/\partial t. \tag{8}$$

Substitution for $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})$ in terms of the Laplacian, and also for $\boldsymbol{\nabla}\times\boldsymbol{E}$ from Maxwell's 3rd equation yields

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{\nabla}^2\boldsymbol{B} = \mu_0\boldsymbol{\nabla}\times\boldsymbol{J}_{\text{total}}^{(e)} - \mu_0\varepsilon_0\partial^2\boldsymbol{B}/\partial t^2. \tag{9}$$

Given that, according to Maxwell's 4th equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, the above equation is further simplified, as follows:

$$\boldsymbol{\nabla}^2\boldsymbol{B} - \frac{\partial^2\boldsymbol{B}}{c^2\partial t^2} = -\mu_0\boldsymbol{\nabla}\times\boldsymbol{J}_{\text{total}}^{(e)}. \tag{10}$$

It is thus seen that we have been able to decouple Maxwell's equations and obtain two 2nd order partial differential equations that relate each of the fields to their corresponding source(s).