Problem 5.50) a) The symmetry of the problem dictates that the potential be a function of the radial distance $\rho = \sqrt{x^2 + y^2}$ from the wire. At an observation point located in the xy-plane at a distance ρ from the origin, we will have

$$\psi(\boldsymbol{r}) = (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}[\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}'$$

$$= (4\pi\varepsilon_o)^{-1} \lim_{z_0\to\infty}\int_{-z_0}^{z_0}\frac{\lambda_o\delta(x')\delta(y')}{\sqrt{(x-x')^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$= \frac{\lambda_o}{2\pi\varepsilon_o}\lim_{z_0\to\infty}\int_0^{z_0}\frac{\mathrm{d}z'}{\sqrt{x^2+y^2+z'^2}} = \frac{\lambda_o}{2\pi\varepsilon_o}\lim_{z_0\to\infty}\ln\left(z'+\sqrt{\rho^2+z'^2}\right)\Big|_{z'=0}^{z_0} \qquad \leftarrow \rho=\sqrt{x^2+y^2}$$

$$= \frac{\lambda_o}{2\pi\varepsilon_o}\left[\lim_{z_0\to\infty}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right) - \ln\rho\right].$$

The first term on the right-hand-side of the above expression is infinitely large, but it does not vary with ρ and may, therefore, be ignored. The scalar potential is thus given by

$$\psi(\boldsymbol{r}) = -(\lambda_o/2\pi\varepsilon_o)\ln\rho.$$

b) Since $\boldsymbol{E} = -\boldsymbol{\nabla}\psi$, we investigate the gradient of the neglected function $\ln(z_0+\sqrt{\rho^2+z_0^2})$ in the limit when $z_0\to\infty$, to see if it has any dependence on the radial distance ρ. We find

$$\frac{\partial}{\partial\rho}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right) = \frac{\rho/\sqrt{\rho^2+z_0^2}}{z_0+\sqrt{\rho^2+z_0^2}} = \frac{\rho/z_0^2}{1+(\rho/z_0)^2+\sqrt{1+(\rho/z_0)^2}}.$$

Thus, for any finite value of ρ, in the limit when $z_0\to\infty$, the denominator of the above expression approaches 2, while the numerator approaches zero. It is thus clear that, for sufficiently large z_0, the contribution to the E-field of $\ln(z_0+\sqrt{\rho^2+z_0^2})$ at any finite radial distance ρ is negligibly small.

c) The calculation of $\boldsymbol{A}(\boldsymbol{r})$ follows essentially the same steps as the above calculation of $\psi(\boldsymbol{r})$. We will have

$$\boldsymbol{A}(\boldsymbol{r}) = (\mu_o/4\pi)\int_{-\infty}^{\infty}[\boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}' = (\mu_o/4\pi)\lim_{z_0\to\infty}\int_{-z_0}^{z_0}\frac{I_o\delta(x')\delta(y')\hat{\boldsymbol{z}}}{\sqrt{(x-x')^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$= \frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\lim_{z_0\to\infty}\int_0^{z_0}\frac{\mathrm{d}z'}{\sqrt{x^2+y^2+z'^2}} = \frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\lim_{z_0\to\infty}\ln\left(z'+\sqrt{\rho^2+z'^2}\right)\Big|_{z'=0}^{z_0}$$

$$= \frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\left[\lim_{z_0\to\infty}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right) - \ln\rho\right].$$

Ignoring the first term on the right-hand-side of the above equation, we find the vector potential of the wire to be $\boldsymbol{A}(\boldsymbol{r}) = -(\mu_o I_o\hat{\boldsymbol{z}}/2\pi)\ln\rho$.