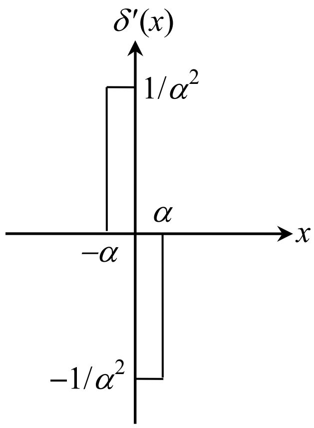
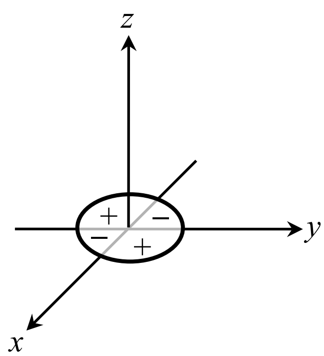
Problem 5.49) a) As shown in figure (a) below, the function $\delta'(x)$ is positive when x is negative, and negative when x is positive. Therefore, the product $\delta'(x)\delta'(y)$ is positive in the first and third quadrants of the xy-plane, and negative in the second and fourth quadrants; see figure (b).

(a)



(b)



b) The charge-density ρ is in units of coulomb/m^3. Since $\delta'(x)$ and $\delta'(y)$ have units of 1/m^2, while the units of $\delta(x)$ are 1/m, we conclude that the units of Q must be coulomb$\cdot$m^2.

c) The scalar potential of the quadrupole may be calculated with the aid of the sifting property of the delta-function and its derivative. We will have

$$\psi(\boldsymbol{r}) = (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}[\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\,\mathrm{d}\boldsymbol{r}'$$

$$= (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q\delta'(x')\delta'(y')\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \quad \leftarrow \boxed{\text{Sifting property of } \delta(z')}$$

$$= \frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta'(x')\delta'(y')}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,\mathrm{d}x'\mathrm{d}y' \quad \leftarrow \boxed{\text{Sifting property of } \delta'(y')}$$

$$= -\frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{y\,\delta'(x')}{[(x-x')^2+y^2+z^2]^{3/2}}\,\mathrm{d}x' \quad \leftarrow \boxed{\text{Sifting property of } \delta'(x')}$$

$$= \frac{3Qxy}{4\pi\varepsilon_o(x^2+y^2+z^2)^{5/2}} = \frac{3Q\sin^2\theta\sin\phi\cos\phi}{4\pi\varepsilon_o r^3} = \frac{3Q\sin^2\theta\sin 2\phi}{8\pi\varepsilon_o r^3}. \quad \leftarrow$$

Spherical coordinates:
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Note that the potential drops with the cube of the distance r from the origin, in contrast with a point-charge, whose potential drops as $1/r$, or a point-dipole, whose potential drops as $1/r^2$.

d) $$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(r,\theta,\phi) = -\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} - \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} - \frac{\partial\psi}{r\sin\theta\partial\phi}\hat{\boldsymbol{\phi}}$$

$$= \frac{9Q\sin^2\theta\sin 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{r}} - \frac{6Q\sin\theta\cos\theta\sin 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{\theta}} - \frac{6Q\sin\theta\cos 2\phi}{8\pi\varepsilon_o r^4}\hat{\boldsymbol{\phi}}$$

$$= \frac{3Q\sin\theta}{8\pi\varepsilon_o r^4}[3\sin\theta\sin(2\phi)\hat{\boldsymbol{r}} - 2\cos\theta\sin(2\phi)\hat{\boldsymbol{\theta}} - 2\cos(2\phi)\hat{\boldsymbol{\phi}}].$$