Problem 5.47)

a) $\nabla\cdot\boldsymbol{D}(\boldsymbol{r},t)=0,$ ← $\rho_{\text{free}}(\boldsymbol{r},t)$ is set to zero.

$\nabla\times\boldsymbol{H}(\boldsymbol{r},t)=\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t,$ ← $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ is set to zero.

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t \quad\rightarrow\quad \nabla\times(\varepsilon_o\boldsymbol{E}+\boldsymbol{P})=-\varepsilon_o(\partial\boldsymbol{M}/\partial t-\varepsilon_o^{-1}\nabla\times\boldsymbol{P})-\mu_o\varepsilon_o\partial\boldsymbol{H}/\partial t$$

$$\rightarrow\quad \nabla\times\boldsymbol{D}(\boldsymbol{r},t)=-\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-\mu_o\varepsilon_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t \qquad \leftarrow \boldsymbol{J}_{\text{bound}}^{(m)}=\partial\boldsymbol{M}/\partial t-\varepsilon_o^{-1}\nabla\times\boldsymbol{P},$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t)=0 \quad\rightarrow\quad \mu_o\nabla\cdot\boldsymbol{H}(\boldsymbol{r},t)=\rho_{\text{bound}}^{(m)} \qquad \leftarrow \rho_{\text{bound}}^{(m)}=-\nabla\cdot\boldsymbol{M}(\boldsymbol{r},t).$$

b) Since Maxwell's 1st equation ensures that $\nabla\cdot\boldsymbol{D}=0$, we define the magnetic vector potential $\boldsymbol{A}^{(m)}(\boldsymbol{r},t)$ such that $\boldsymbol{D}(\boldsymbol{r},t)=-\nabla\times\boldsymbol{A}^{(m)}(\boldsymbol{r},t)$. Substitution into Maxwell's 2nd equation yields

$$\nabla\times[\boldsymbol{H}(\boldsymbol{r},t)+\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t]=0 \quad\rightarrow\quad \boldsymbol{H}(\boldsymbol{r},t)+\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t=-\nabla\psi^{(m)}(\boldsymbol{r},t).$$ ← Because $\nabla\times\nabla\psi^{(m)}=0.$

c) Using the above magnetic potentials, Maxwell's 3rd equation may be written as follows:

$$\nabla\times[-\nabla\times\boldsymbol{A}^{(m)}(\boldsymbol{r},t)]=-\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-\mu_o\varepsilon_o(\partial/\partial t)[-\nabla\psi^{(m)}(\boldsymbol{r},t)-\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t]$$

$$\rightarrow\quad \nabla[\nabla\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)]-\nabla^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)=\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-(1/c^2)\nabla[\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t]-(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2$$

$$\rightarrow\quad \nabla[\nabla\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t]-\nabla^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2=\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}.$$

Clearly, eliminating $\psi^{(m)}$ from the above equation requires the same gauge for the magnetic potentials as the Lorenz gauge that was defined for the electric potentials, that is,

$$\nabla\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t=0.$$ ← Equivalent of Lorenz gauge

d) Maxwell's 3rd equation in conjunction with the Lorenz-equivalent gauge now becomes

$$\nabla^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)-(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2=-\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}.$$

This wave equation for the magnetic vector potential, having $\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}$ for the source term, is the counterpart of the equation for the standard vector potential, where the source term is $\mu_o\boldsymbol{J}_{\text{total}}^{(e)}$.

The wave equation for $\psi^{(m)}(\boldsymbol{r},t)$ could similarly be derived by substituting into Maxwell's 4th equation the expressions that relate $\boldsymbol{D}$ and $\boldsymbol{H}$ to $\boldsymbol{A}^{(m)}$ and $\psi^{(m)}$. The final result, after taking account of the Lorenz-equivalent gauge, is

$$\mu_o\nabla\cdot[-\nabla\psi^{(m)}-\partial\boldsymbol{A}^{(m)}/\partial t]=\rho_{\text{bound}}^{(m)} \quad\rightarrow\quad \nabla^2\psi^{(m)}(\boldsymbol{r},t)-(1/c^2)\partial^2\psi^{(m)}(\boldsymbol{r},t)/\partial t^2=-\mu_o^{-1}\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t).$$

Once again we have an equation similar to the standard wave equation for the (electric) scalar potential, except that the source term here is $\mu_o^{-1}\rho_{\text{bound}}^{(m)}$, rather than $\varepsilon_o^{-1}\rho_{\text{total}}^{(e)}$.

e) By analogy with the standard wave equation, we write the solutions to the preceding wave equations for magnetic potentials as follows:

$$\psi^{(m)}(\boldsymbol{r},t) = (1/4\pi\mu_o)\int_{-\infty}^{\infty} \frac{\rho_{\text{bound}}^{(m)}(\boldsymbol{r}', t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}'.$$

$$\boldsymbol{A}^{(m)}(\boldsymbol{r},t) = (\varepsilon_o/4\pi)\int_{-\infty}^{\infty} \frac{\boldsymbol{J}_{\text{bound}}^{(m)}(\boldsymbol{r}', t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}';$$
