Problem 5.45)

a) $\boldsymbol{P}(\boldsymbol{r},t) = p_0\delta(x)\delta(y)\hat{\boldsymbol{z}} \quad \rightarrow \quad \rho^{(\mathrm{e})}_{\mathrm{bound}} = -\boldsymbol{\nabla}\cdot\boldsymbol{P} = -\frac{\partial P_z}{\partial z} = 0$ and $\boldsymbol{J}^{(\mathrm{e})}_{\mathrm{bound}} = \frac{\partial \boldsymbol{P}}{\partial t} = 0.$

b) Since the wire has no charge and no current, both its scalar and vector potentials must be zero.

c) In the absence of charge and current, there will be no electric and no magnetic fields.

d) $\boldsymbol{M}(\boldsymbol{r},t) = m_0\delta(x)\delta(y)\hat{\boldsymbol{z}} \quad \rightarrow \quad \boldsymbol{J}^{(\mathrm{e})}_{\mathrm{bound}} = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M} = \mu_0^{-1}m_0[\delta(x)\delta'(y)\hat{\boldsymbol{x}} - \delta'(x)\delta(y)\hat{\boldsymbol{y}}].$

Since the wire has no electric charges, its scalar potential is zero, that is, $\psi(\boldsymbol{r},t) = 0$. As for vector potential, since the electric current is constant in time, the wire's vector potential will be time-independent. We thus write

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint_{-\infty}^{\infty}\frac{\boldsymbol{J}^{(\mathrm{e})}_{\mathrm{bound}}(\tilde{\boldsymbol{r}})}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}d\tilde{x}d\tilde{y}d\tilde{z} = \frac{m_0}{4\pi}\iiint_{-\infty}^{\infty}\frac{\delta(\tilde{x})\delta'(\tilde{y})\hat{\boldsymbol{x}} - \delta'(\tilde{x})\delta(\tilde{y})\hat{\boldsymbol{y}}}{\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}}d\tilde{x}d\tilde{y}d\tilde{z}$$

Use sifting properties of $\delta(\cdot)$ and $\delta'(\cdot)$. →

$$= \frac{m_0}{4\pi}\int_{-\infty}^{\infty}\frac{(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})}{[x^2+y^2+(z-\tilde{z})^2]^{3/2}}d\tilde{z} = \frac{m_0(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})}{4\pi(x^2+y^2)}\int_{-\infty}^{\infty}\frac{d\tilde{z}}{\sqrt{x^2+y^2}\,\{1+[(z-\tilde{z})/\sqrt{x^2+y^2}\,]^2\}^{3/2}}$$

Change variable to $\zeta = \frac{z-\tilde{z}}{\sqrt{x^2+y^2}}$. →

$$= \frac{m_0(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})}{4\pi(x^2+y^2)}\int_{-\infty}^{\infty}\frac{d\zeta}{(1+\zeta^2)^{3/2}} = \frac{m_0\hat{\boldsymbol{\phi}}}{4\pi\sqrt{x^2+y^2}}\left.\frac{\zeta}{\sqrt{1+\zeta^2}}\right|_{-\infty}^{\infty} = \frac{m_0\hat{\boldsymbol{\phi}}}{2\pi\rho}$$

← Switch to cylindrical coordinates (ρ,ϕ,z).

Considering that the scalar potential is zero and the vector potential is time-independent, the E-field surrounding the magnetic wire is found to be zero, that is, $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t = 0$. As for the magnetic field, the curl of $\boldsymbol{A}(\boldsymbol{r})$ can be readily calculated in cylindrical coordinates and seen to be zero everywhere, except, along the z-axis, where $\boldsymbol{A}(\boldsymbol{r})$ is singular. Using the definition of the curl operator in the vicinity of the z-axis, we find that $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A} = m_0\delta(x)\delta(y)\hat{\boldsymbol{z}}$. This, of course, is a consequence of the fact that, by definition, $\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M}$, and that, in the absence of magnetic charges, i.e., $\rho^{(\mathrm{m})}_{\mathrm{bound}} = -\boldsymbol{\nabla}\cdot\boldsymbol{M} = 0$, the H-field everywhere is zero. Consequently, the B-field exists only within the wire, where $\boldsymbol{B} = \boldsymbol{M} = m_0\delta(x)\delta(y)\hat{\boldsymbol{z}}$.

Digression: An alternative means of calculating the magnetic wire's vector potential is the Fourier method, namely,

$$\boldsymbol{M}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\boldsymbol{M}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{r}dt$$

$$= \int_{-\infty}^{\infty}m_0\delta(x)\delta(y)\hat{\boldsymbol{z}}\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{r}dt = (2\pi)^2 m_0\delta(k_z)\delta(\omega)\hat{\boldsymbol{z}}.$$

$$\boldsymbol{J}^{(\mathrm{e})}_{\mathrm{bound}}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega) = (2\pi)^2\mu_0^{-1}m_0\delta(k_z)\delta(\omega)(\mathrm{i}\boldsymbol{k}\times\hat{\boldsymbol{z}}).$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\mu_0\boldsymbol{J}^{(\mathrm{e})}_{\mathrm{bound}}(\boldsymbol{k},\omega)}{k^2-(\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{k}d\omega$$

$$= -\left(\frac{\mathrm{i}m_0}{4\pi^2}\right)\hat{\boldsymbol{z}}\times\int_{-\infty}^{\infty}\frac{\boldsymbol{k}\,\delta(k_z)\delta(\omega)}{k^2-(\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{k}d\omega$$

$$= -\left(\frac{\mathrm{i}m_0}{4\pi^2}\right)\hat{\boldsymbol{z}}\times\int_{-\infty}^{\infty}\frac{\boldsymbol{k}_\parallel\exp(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{\rho})}{k_\parallel^2}d\boldsymbol{k}_\parallel$$

← $\boldsymbol{k}_\parallel = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, $\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$

$$= -\left(\frac{\mathrm{i}m_0}{4\pi^2}\right)\hat{\boldsymbol{z}} \times \int_{k_\parallel=0}^{\infty}\int_{\varphi=0}^{2\pi} \frac{(k_\parallel \cos\varphi)\,\hat{\boldsymbol{\rho}}\,\exp(\mathrm{i}k_\parallel \rho\cos\varphi)}{k_\parallel^2} k_\parallel d\varphi dk_\parallel$$

$$= -\left(\frac{\mathrm{i}m_0}{4\pi^2}\right)(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\rho}}) \int_{k_\parallel=0}^{\infty} \left[\int_{\varphi=0}^{2\pi} \cos\varphi \exp(\mathrm{i}k_\parallel \rho\cos\varphi) d\varphi\right] dk_\parallel$$

$$= \frac{m_0\hat{\boldsymbol{\phi}}}{2\pi}\int_0^{\infty} J_1(k_\parallel \rho) dk_\parallel = \frac{m_0\hat{\boldsymbol{\phi}}}{2\pi\rho}.$$

← $J_1(\cdot)$ is Bessel function of first kind, 1st order.

This, of course, is the same solution for the vector potential as was obtained before.
