Problem 5.44) The magnetization distribution $\boldsymbol{M}(\boldsymbol{r},t)$ does not produce any (bound) electrical charges. Therefore $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)=0$. The absence of electrical charge implies that the scalar potential (in the Lorenz gauge) is also absent in this problem, that is, $\psi(\boldsymbol{r},t)=0$.

Since this is a magnetostatic problem (i.e., the magnetization is time-independent), the bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)$ and, consequently, the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$, will also be time-independent. As a result, we will have $\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi(\boldsymbol{r},t)-\partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t=0$.

a)
$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{\nabla}\times[m_0\delta(x)\delta(y)\hat{\boldsymbol{z}}]$$
$$=\mu_0^{-1}m_0[\delta(x)\delta'(y)\hat{\boldsymbol{x}}-\delta'(x)\delta(y)\hat{\boldsymbol{y}}]$$

b) The symmetry of the problem allows us to choose the observation point $\boldsymbol{r}$ as an arbitrary point in the xy-plane, where $z=0$. In other words, $\boldsymbol{r}=x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}$. Also, since the current-density is time independent, the term $t-|\boldsymbol{r}-\boldsymbol{r}'|/c$ can be dropped from the vector potential formula. We will have

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu_0}{4\pi}\iiint_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'=\frac{m_0}{4\pi}\iiint_{-\infty}^{\infty}\frac{\delta(x')\delta'(y')\hat{\boldsymbol{x}}-\delta'(x')\delta(y')\hat{\boldsymbol{y}}}{\sqrt{(x-x')^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

Sifting property of $\delta(x')$ and $\delta(y')$ →
$$=\frac{m_0}{4\pi}\left[\hat{\boldsymbol{x}}\iint_{-\infty}^{\infty}\frac{\delta'(y')}{\sqrt{x^2+(y-y')^2+z'^2}}\mathrm{d}y'\mathrm{d}z'-\hat{\boldsymbol{y}}\iint_{-\infty}^{\infty}\frac{\delta'(x')}{\sqrt{(x-x')^2+y^2+z'^2}}\mathrm{d}x'\mathrm{d}z'\right]$$

Sifting property of $\delta'(x')$ and $\delta'(y')$ →
$$=\frac{m_0}{4\pi}\left[-\hat{\boldsymbol{x}}\int_{-\infty}^{\infty}\frac{y}{(x^2+y^2+z'^2)^{3/2}}\mathrm{d}z'+\hat{\boldsymbol{y}}\int_{-\infty}^{\infty}\frac{x}{(x^2+y^2+z'^2)^{3/2}}\mathrm{d}z'\right]$$

$$=\frac{m_0}{2\pi}(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})\int_0^{\infty}\frac{\mathrm{d}z'}{(x^2+y^2+z'^2)^{3/2}}=\frac{m_0}{2\pi}(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})\left.\frac{z'}{(x^2+y^2)\sqrt{x^2+y^2+z'^2}}\right|_{z'=0}^{\infty}$$

$$=\frac{m_0}{2\pi}\left(\frac{x\hat{\boldsymbol{y}}-y\hat{\boldsymbol{x}}}{x^2+y^2}\right)=\left(\frac{m_0}{2\pi}\right)\frac{\hat{\boldsymbol{z}}\times(x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}})}{x^2+y^2}=\left(\frac{m_0}{2\pi}\right)\frac{\hat{\boldsymbol{z}}\times\boldsymbol{r}}{r^2}=\frac{m_0\hat{\boldsymbol{\phi}}}{2\pi r}.$$ ← cylindrical coordinates

c)
$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)=-\frac{\partial A_\phi}{\partial z}\hat{\boldsymbol{r}}+\frac{\partial(rA_\phi)}{r\partial r}\hat{\boldsymbol{z}}=\frac{\partial(m_0/2\pi)}{r\partial r}\hat{\boldsymbol{z}}=0.$$

The B-field, and also the H-field, are thus seen to be zero everywhere outside the wire—even though the vector potential is not zero. Note that on the z-axis itself, the curl of $\boldsymbol{A}(\boldsymbol{r})$ is *not* zero. Using the definition of Curl ($\boldsymbol{\nabla}\times$) as the integral of $\boldsymbol{A}(\boldsymbol{r})$ around a small loop, normalized by the loop area, the B-field inside the wire is readily found to be $m_0\delta(x)\delta(y)\hat{\boldsymbol{z}}$. This is simply the magnetization $\boldsymbol{M}(\boldsymbol{r})$ of the wire. Considering that $\boldsymbol{B}=\mu_0\boldsymbol{H}+\boldsymbol{M}$, we conclude that the H-field inside the wire is zero as well.