Problem 5.43)

a) $\rho(\boldsymbol{r},t) = \lambda_0\delta(x)\delta(y)\text{Rect}\left(\frac{z}{2L}\right)$.

b) $$\psi(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\rho(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}' = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda_0\delta(x')\delta(y')\text{Rect}(z'/2L)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\int_{z'=-L}^{L}\frac{dz'}{\sqrt{x^2+y^2+(z'-z)^2}} = \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left[(z'-z)+\sqrt{x^2+y^2+(z'-z)^2}\right]\Big|_{z'=-L}^{L}$$

$$= \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left[\frac{\sqrt{r^2+(L-z)^2}+(L-z)}{\sqrt{r^2+(L+z)^2}-(L+z)}\right] = \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left\{\frac{\left[\sqrt{r^2+(L-z)^2}+(L-z)\right]\left[\sqrt{r^2+(L+z)^2}+(L+z)\right]}{r^2}\right\}.$$

c) Introducing the normalized parameters $\tilde{r} = r/L$ and $\tilde{z} = z/L$, the above equation may be written as follows:

$$\psi(\boldsymbol{r},t) = -\frac{\lambda_0\ln r}{2\pi\varepsilon_0}+\frac{\lambda_0\ln L}{2\pi\varepsilon_0}+\frac{\lambda_0}{4\pi\varepsilon_0}\ln\left\{\left[\sqrt{(1-\tilde{z})^2+\tilde{r}^2}+(1-\tilde{z})\right]\left[\sqrt{(1+\tilde{z})^2+\tilde{r}^2}+(1+\tilde{z})\right]\right\}.$$

In the limit when $L\to\infty$, both $\tilde{r}$ and $\tilde{z}$ approach zero, and the above equation becomes

$$\psi(\boldsymbol{r},t) = \frac{\lambda_0\ln(2L)}{2\pi\varepsilon_0}-\frac{\lambda_0\ln r}{2\pi\varepsilon_0}.$$

The large constant containing $\ln(2L)$ in the above expression does *not* contribute to the gradient of the scalar potential. Therefore, the E-field of the infinitely-long rod is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi = -\left(\frac{\partial\psi}{\partial r}\right)\hat{\boldsymbol{r}} = \frac{\lambda_0}{2\pi\varepsilon_0 r}\hat{\boldsymbol{r}}.$$

d) The Fourier transform of the charge-density distribution is given by

$$\rho(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\rho(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{r}dt$$

$$= 2\pi\delta(\omega)\lambda_0\int_{-L}^{L}\exp(-\mathrm{i}k_z z)\,dz = 4\pi\lambda_0\delta(\omega)\sin(Lk_z)/k_z.$$

Since the Fourier-transformed scalar potential is $\psi(\boldsymbol{k},\omega) = \varepsilon_0^{-1}\rho(\boldsymbol{k},\omega)/[k^2-(\omega/c)^2]$, its inverse transform may now be evaluated as follows:

$$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{k}d\omega$$

$$= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{-\infty}^{\infty}\frac{\sin(Lk_z)}{k_z\,k^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k}$$

$$= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{-\infty}^{\infty}\frac{\sin(Lk_z)\exp(\mathrm{i}k_z z)}{k_z(k_x^2+k_y^2+k_z^2)}\exp\left[\mathrm{i}(k_x x+k_y y)\right]dk_x dk_y dk_z$$

← Define $\boldsymbol{k}_\parallel = k_x\hat{\boldsymbol{x}}+k_y\hat{\boldsymbol{y}}$ and $\boldsymbol{r}_\parallel = x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}$.

$$= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{k_z=-\infty}^{\infty}\frac{\sin(Lk_z)[\cos(k_z z)+\mathrm{i}\sin(k_z z)]}{k_z}\int_{k_\parallel=0}^{\infty}\frac{1}{k_\parallel^2+k_z^2}\int_{\varphi=0}^{2\pi}\exp(\mathrm{i}k_\parallel r_\parallel\cos\varphi)k_\parallel d\varphi\,dk_\parallel dk_z$$

$$= \frac{\lambda_0}{(2\pi)^2 \varepsilon_0} \int_{-\infty}^{\infty} \frac{\{\sin[k_z(L+z)] + \sin[k_z(L-z)]\} + \mathrm{i}\{\cos[k_z(L-z)] - \cos[k_z(L+z)]\}}{k_z} \int_{k_\parallel = 0}^{\infty} \frac{k_\parallel J_0(k_\parallel r_\parallel)}{k_\parallel^2 + k_z^2} dk_\parallel dk_z$$

$$= \frac{\lambda_0}{(2\pi)^2 \varepsilon_0} \int_{-\infty}^{\infty} \frac{\{\sin[k_z(L+z)] + \sin[k_z(L-z)]\} - \mathrm{i}\{\cos[k_z(L+z)] - \cos[k_z(L-z)]\}}{k_z} K_0(r_\parallel |k_z|) dk_z$$

The terms of the integrand that contain cosines are odd functions of k_z; therefore, their integrals vanish.

$$= \frac{\lambda_0}{(2\pi)^2 \varepsilon_0} \int_{-\infty}^{\infty} k_z^{-1} \{\sin[k_z(L+z)] + \sin[k_z(L-z)]\} K_0(r_\parallel |k_z|) dk_z$$

$$= \frac{\pi \lambda_0}{(2\pi)^2 \varepsilon_0} \left\{ \ln\left[\left(\frac{L+z}{r_\parallel}\right) + \sqrt{1 + \left(\frac{L+z}{r_\parallel}\right)^2} \right] + \ln\left[\left(\frac{L-z}{r_\parallel}\right) + \sqrt{1 + \left(\frac{L-z}{r_\parallel}\right)^2} \right] \right\}$$

$$= -\frac{\lambda_0 \ln r_\parallel}{2\pi\varepsilon_0} + \frac{\lambda_0}{4\pi\varepsilon_0} \ln\left\{ \left[(L+z) + \sqrt{r_\parallel^2 + (L+z)^2} \right] \left[(L-z) + \sqrt{r_\parallel^2 + (L-z)^2} \right] \right\}.$$

This result is identical with that obtained in part (b), which was obtained using direct evaluation in the spacetime domain.