Problem 5.42) Without doing any calculations, it is obvious that an H-field that is uniform inside the rod, and zero outside, is the solution to the problem, because such a field satisfies the relevant Maxwell's equations ($\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B}=\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}=0$) everywhere. Nevertheless, we proceed to offer a formal proof.

The infinite length of the rod allows us to pick the observation point $\boldsymbol{r}$ in the xy-plane, where $z=0$. Take a point $\boldsymbol{r}'$ within the rod, and use the Biot-Savart law to calculate the H-field at the observation point $\boldsymbol{r}$ by integrating over all the volume elements of the rod, that is,

$$\boldsymbol{H}(\boldsymbol{r})=\frac{1}{4\pi}\iiint_{\substack{\text{rod's}\\ \text{volume}}}\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}v'$$

Considering that the surface current $\boldsymbol{J}_s$ is tangential to the surface element $\mathrm{d}s'$ and also perpendicular to the z-axis, we can write $\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}v'=J_s\hat{\boldsymbol{z}}\times\mathrm{d}\boldsymbol{s}'$, where, as usual, $\mathrm{d}\boldsymbol{s}'$ is taken to be perpendicular to the surface element $\mathrm{d}s'$ centered at $\boldsymbol{r}'$, and pointing outward. We will have

$$\boldsymbol{H}(\boldsymbol{r})=(J_s/4\pi)\iint_{\substack{\text{rod's}\\ \text{surface}}}\frac{(\hat{\boldsymbol{z}}\times\mathrm{d}\boldsymbol{s}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}=(J_s/4\pi)\iint_{\substack{\text{rod's}\\ \text{surface}}}\frac{[(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{\boldsymbol{z}}]\mathrm{d}\boldsymbol{s}'-[(\boldsymbol{r}-\boldsymbol{r}')\cdot\mathrm{d}\boldsymbol{s}']\hat{\boldsymbol{z}}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

The vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ has been used in the above equation. Now, the first term in the numerator of the integrand reduces to

$$[(\boldsymbol{r}-\boldsymbol{r}')\cdot\hat{\boldsymbol{z}}]\mathrm{d}\boldsymbol{s}'=\{[(x-x')\hat{\boldsymbol{x}}+(y-y')\hat{\boldsymbol{y}}-z'\hat{\boldsymbol{z}}]\cdot\hat{\boldsymbol{z}}\}\mathrm{d}\boldsymbol{s}'=-z'\mathrm{d}\boldsymbol{s}'.$$

Consequently, the first term of the integrand, $-z'\mathrm{d}\boldsymbol{s}'/|\boldsymbol{r}-\boldsymbol{r}'|^3$, being an odd function of z', integrates to zero. As for the second term, denoting by $\mathrm{d}\Omega'$ the solid angle subtended by $\mathrm{d}s'$ at the observation point $\boldsymbol{r}$, we write

$$\boldsymbol{H}(\boldsymbol{r})=(J_s\hat{\boldsymbol{z}}/4\pi)\iint_{\substack{\text{rod's}\\ \text{surface}}}\frac{\boldsymbol{r}'-\boldsymbol{r}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\cdot\mathrm{d}\boldsymbol{s}'=(J_s\hat{\boldsymbol{z}}/4\pi)\iint_{\text{rod's surface}}\mathrm{d}\Omega'.$$

It is seen that, if the observation point $\boldsymbol{r}$ happens to be *inside* the rod, the solid angles subtended by the various surface elements will cover the entire 3D space, thus integrating to 4π and yielding $\boldsymbol{H}(\boldsymbol{r})=J_s\hat{\boldsymbol{z}}$. In contrast, when the observation point is *outside* the rod, the sign of $\mathrm{d}\Omega'$ will depend on whether $\mathrm{d}\boldsymbol{s}'$ is pointing toward or away from the observation point. It is not difficult to see that, in this case, the contributions to the integral of the front facet of the rod cancel those of the rear facet (as seen from the observation point), yielding $\boldsymbol{H}(\boldsymbol{r})=0$.