Problem 41) The surface charge-density as a function of time is given by

$$\sigma_s(t) = \frac{Q}{4\pi r^2(t)} = \frac{R^2\sigma_{so}}{[R + a_o\sin(\omega_o t)]^2}. \tag{1}$$

When the sphere's surface expands/contracts at the velocity $\boldsymbol{V}_s(t) = \dot{r}_s(t) = a_o\,\omega_o\cos(\omega_o t)\hat{\boldsymbol{r}}$, it produces the current-density

$$\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = \sigma_s(t)\,a_o\omega_o\cos(\omega_o t)\,\delta[r - r_s(t)]\hat{\boldsymbol{r}}. \tag{2}$$

Here $\delta(\cdot)$ is a Dirac delta-function of the radial coordinate r.

a) The E-field must be oriented along the radial direction $\hat{\boldsymbol{r}}$, as symmetry of the problem prohibits the field from having $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ components. Applying the integral form of Gauss's law to a (concentric) sphere of radius r yields

$$\boldsymbol{E}(\boldsymbol{r},t) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_o r^2}; & r > r_s(t), \\ 0; & r < r_s(t). \end{cases} \tag{3}$$

b) The vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ must also be oriented along the radial direction, as the current density at the surface of the charged sphere is always radial, and as the contributions along $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ of the various current elements to $\boldsymbol{A}(\boldsymbol{r},t)$ evaluated at a fixed observation point cancel out. Moreover, symmetry does *not* allow the remaining (radial) component, A_r, to depend on the angular coordinates θ and ϕ. Thus $\boldsymbol{A}(\boldsymbol{r},t)$ can only be a function of the radial distance r. The curl of such a vector potential, $\boldsymbol{\nabla}\times A_r(r,t)\hat{\boldsymbol{r}}$, is readily seen to be zero. Consequently, the magnetic field everywhere inside and outside the sphere must vanish.

One may also argue that the scalar potential $\psi(\boldsymbol{r},t)$ must be similarly independent of θ and ϕ, and that, therefore, the E-field, given by

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t = -[\partial\psi(r,t)/\partial r + \partial A_r(r,t)/\partial t]\hat{\boldsymbol{r}}, \tag{4}$$

is independent of θ and ϕ, and is aligned with $\hat{\boldsymbol{r}}$. This, of course, is the same conclusion reached in part (a) based on a direct argument from symmetry.

c) The E-field found in part (a) obviously satisfies Maxwell's 1st equation. It is also easy to see from Eqs.(1) and (3) that, at the surface of the charged sphere, the discontinuity of the E-field is equal to $\sigma_s(t)/\varepsilon_o$.

To satisfy Maxwell's 2nd equation, since $\boldsymbol{H}(\boldsymbol{r},t) = 0$ and, therefore, $\boldsymbol{\nabla}\times\boldsymbol{H} = 0$ everywhere, we must show that $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \varepsilon_o\partial\boldsymbol{E}(\boldsymbol{r},t)/\partial t = 0$. The only points where $\boldsymbol{E}$ varies with time are those at the surface of the charged sphere, where the D-field jumps discontinuously between zero and $\sigma_s(t)\hat{\boldsymbol{r}}$; see Eqs.(1) and (3). At these points the local $\varepsilon_o\partial\boldsymbol{E}(\boldsymbol{r},t)/\partial t$ is a Dirac delta-function, namely, $\pm\sigma_s(t)\delta(t - t_o)\hat{\boldsymbol{r}}$, where t_o is the instant of switching. Assuming the surface charge is confined to a layer of vanishingly small thickness τ, the time needed for the D-field to switch between zero and $\sigma_s(t_o)\hat{\boldsymbol{r}}$ will be $\tau/V_s(t_o)=\tau/|a_o\omega_o\cos(\omega_o t_o)|$, which defines the "width" of $\delta(t-t_o)$. Now, the current density of Eq.(2) is also expressed in terms of a delta-function, $\delta[r-r_s(t)]$, but this is a delta-function of width τ defined on the radial coordinate r. It is a well-known fact that, in the vicinity of t_o where $f(t_o)=0$, the function $\delta[f(t)]$ may be written as $\delta(t-t_o)/|f'(t_o)|$. Therefore, $\delta[r-r_s(t)]=\delta(t-t_o)/|a_o\omega_o\cos(\omega_o t)|$. Substitution into Eq.(2) now confirms that $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\varepsilon_o\partial\boldsymbol{E}(\boldsymbol{r},t)/\partial t=0$, as required.

The 3rd and 4th of Maxwell's equations are easily seen to be satisfied as well, because $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=0$ and also $\boldsymbol{B}(\boldsymbol{r},t)=0$.
