Problem 40)

a) Electric dipole:

$$\vec{E}(\vec{r},t)=\frac{Z_0 I_0 d}{4\pi}\left\{\left[\frac{1}{r^2}\sin(\omega t-k_0 r)-\frac{(\lambda_0/2\pi)}{r^3}\cos(\omega t-k_0 r)\right](2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})+\frac{k_0}{r}\sin\theta\cos(\omega t-k_0 r)\,\hat{\theta}\right\}$$

$$\vec{H}(\vec{r},t)=\frac{I_0 d}{4\pi}\sin\theta\left\{\frac{1}{r^2}\sin(\omega t-k_0 r)+\frac{k_0}{r}\cos(\omega t-k_0 r)\right\}\hat{\phi}$$

$$\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=Z_0\left(\frac{I_0 d}{4\pi}\right)^2\sin\theta\left\{\left[\frac{\sin(\omega t-k_0 r)}{r^2}-\frac{\cos(\omega t-k_0 r)}{k_0 r^3}\right]\left[\frac{\sin(\omega t-k_0 r)}{r^2}+\frac{k_0\cos(\omega t-k_0 r)}{r}\right](-2\cos\theta\,\hat{\theta}+\sin\theta\,\hat{r})+\frac{k_0}{r}\sin\theta\cos(\omega t-k_0 r)\left[\frac{\sin(\omega t-k_0 r)}{r^2}+\frac{k_0\cos(\omega t-k_0 r)}{r}\right]\hat{r}\right\}\Rightarrow$$

$$\vec{S}(\vec{r},t)=Z_0\left(\frac{I_0 d}{4\pi}\right)^2\sin\theta\left\{\left[\frac{\sin^2(\omega t-k_0 r)}{r^4}-\frac{\cos^2(\omega t-k_0 r)}{r^4}+\frac{1}{2}\sin 2(\omega t-k_0 r)\left(\frac{k_0}{r^3}-\frac{1}{k_0 r^5}\right)\right](\sin\theta\,\hat{r}-2\cos\theta\,\hat{\theta})+\left[\frac{k_0\sin\theta}{2r^3}\sin 2(\omega t-k_0 r)+\sin\theta\left(\frac{k_0}{r}\right)^2\cos^2(\omega t-k_0 r)\right]\hat{r}\right\}\Rightarrow$$

$$\vec{S}(\vec{r},t)=Z_0\left(\frac{I_0 d}{4\pi}\right)^2\left\{\left[\left(\frac{k_0}{2r^3}-\frac{1}{2k_0 r^5}\right)\sin(2\omega t-2k_0 r)-\frac{1}{r^4}\cos(2\omega t-2k_0 r)\right](\sin^2\theta\,\hat{r}-\sin 2\theta\,\hat{\theta})+\left[\frac{k_0\sin^2\theta}{2r^3}\sin(2\omega t-2k_0 r)+\left(\frac{k_0\sin\theta}{r}\right)^2\cos^2(\omega t-k_0 r)\right]\hat{r}\right\}$$

Here $k_0=2\pi/\lambda_0=2\pi f/c=\omega/c$ and $f=1/T$, where T is the period of oscillations. The time-averaged Poynting vector is the average of $\vec{S}(\vec{r},t)$ over one period of oscillations; that is,

$$\langle\vec{S}(\vec{r},t)\rangle=\frac{1}{T}\int_0^T\vec{S}(\vec{r},t)\,dt.$$

Since the time-averages of $\sin(2\omega t-2k_0 r)$ and $\cos(2\omega t-2k_0 r)$ are zero, and the time-average of $\cos^2(\omega t-k_0 r)=\frac{1}{2}+\frac{1}{2}\cos(2\omega t-2k_0 r)$ is equal to $1/2$, we'll have:

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$$\langle \vec{S}(\vec{r},t)\rangle = Z_0\left(\frac{I_0 d}{4\pi}\right)^2\left(\frac{k_0 \sin\theta}{r}\right)^2 \langle \cos^2(\omega t - k_0 r)\rangle\,\hat{r} \Rightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{8} Z_0 I_0^2 (d/\lambda_0)^2 \frac{\sin^2\theta}{r^2}\,\hat{r}$$

If the total radiated power is desired, the above time-averaged Poynting vector may be integrated over the surface of a sphere of fixed radius r. The result is:

$$\text{Total radiated power} = \int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta\, \langle S(\vec{r},t)\rangle\, d\theta = \frac{\pi Z_0}{4}(I_0 d/\lambda_0)^2 \int_0^{\pi} \sin^3\theta\, d\theta$$

where $\int_0^{\pi} \sin^3\theta\, d\theta = \int_0^{\pi} \sin\theta(1-\cos^2\theta)\,d\theta = \int_0^{\pi}\sin\theta\, d\theta - \int_0^{\pi}\sin\theta\cos^2\theta\, d\theta = -\cos\theta\Big|_0^{\pi}$

$+\frac{1}{3}\cos^3\theta\Big|_0^{\pi} = 2 - \frac{2}{3} = 4/3$. Therefore,

$$\text{Total radiated power} = \frac{1}{3}\pi Z_0 (I_0 d/\lambda_0)^2$$

Since the radiation fields for a dipole that we have obtained are not valid in the immediate vicinity of the dipole itself (i.e., in the near near field) we cannot calculate the $\vec{E}$-field that must be applied to the short dipole antenna in order to drive the current $I_0 \sin(\omega t)$ within the dipole.

b) Small loop of current (magnetic dipole): We saw in HW #6, Prob. 4:

$$\vec{E}(\vec{r},t) = \frac{Z_0 m_0 k_0}{4\pi}\sin\theta\left\{\frac{1}{r^2}\sin(\omega t - k_0 r) + \frac{k_0}{r}\cos(\omega t - k_0 r)\right\}\hat{\phi}$$

$$\vec{H}(\vec{r},t) = -\frac{m_0 k_0}{4\pi}\left\{\left[\frac{1}{r^2}\sin(\omega t - k_0 r) - \frac{(\lambda_0/2\pi)}{r^3}\cos(\omega t - k_0 r)\right](2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) + \frac{k_0}{r}\sin\theta\cos(\omega t - k_0 r)\hat{\theta}\right\}$$

These fields are identical with those of an electric dipole studied in part (a) if the roles of $\vec{E}$ and $\vec{H}$ are reversed, and if $m_0 k_0$ is substituted for $I_0 d$. Note that the units of $m_o k_o$ and $I_o d$ are the same, namely, Amp-meter. Again the expressions for $\vec{E}$ and $\vec{H}$ are not valid in the near-near-field, so we cannot use them to determine the necessary voltage to drive the current around the loop.

C) Hollow Cylinder (Radius = R): We saw in HW#7, Problem 2:

$$\vec{E}(\vec{r},t) = \frac{1}{4} Z_o I_o k_o \begin{cases} J_o(k_o R)\left[Y_o(k_o\rho)\cos\omega t - J_o(k_o\rho)\sin\omega t\right]\hat{z}\,; & \rho \geq R \\ \left[Y_o(k_o R)\cos\omega t - J_o(k_o R)\sin\omega t\right] J_o(k_o\rho)\hat{z}\,; & \rho \leq R \end{cases}$$

$$\vec{H}(\vec{r},t) = -\frac{1}{4} I_o k_o \begin{cases} J_o(k_o R)\left[Y_1(k_o\rho)\sin\omega t + J_1(k_o\rho)\cos\omega t\right]\hat{\phi}\,; & \rho \geq R \\ \left[Y_o(k_o R)\sin\omega t + J_o(k_o R)\cos\omega t\right] J_1(k_o\rho)\hat{\phi}\,; & \rho < R \end{cases}$$

First we compute the Poynting vector inside the cylinder ($\rho < R$):

$$\vec{S}(\rho,t) = \vec{E}\times\vec{H} = \frac{1}{16} Z_o (I_o k_o)^2 \Big[Y_o^2(k_o R)\sin\omega t\cos\omega t - J_o^2(k_o R)\sin\omega t\cos\omega t$$

$$+ Y_o(k_o R) J_o(k_o R)\cos^2\omega t - Y_o(k_o R) J_o(k_o R)\sin^2\omega t\Big] J_o(k_o\rho) J_1(k_o\rho)\,\hat{\rho} \Rightarrow$$

$$\vec{S}(\rho,t) = \frac{Z_o}{32}(I_o k_o)^2 \left\{\left[Y_o^2(k_o R) - J_o^2(k_o R)\right]\sin(2\omega t) + 2\,Y_o(k_o R) J_o(k_o R)\cos(2\omega t)\right\} \times J_o(k_o\rho) J_1(k_o\rho)\,\hat{\rho}\,; \quad \rho < R$$

Since the time averages of $\sin(2\omega t)$ and $\cos(2\omega t)$ are zero, we'll have:

$$\langle \vec{S}(\rho,t)\rangle = 0\,; \quad \rho < R$$

The Poynting vector outside the cylinder is given by:

$$\vec{S}(\rho,t) = \frac{1}{16} Z_0 (I_0 k_0)^2 J_0^2(k_0 R) \left\{ \frac{1}{2}\left[Y_0(k_0\rho) Y_1(k_0\rho) - J_0(k_0\rho) J_1(k_0\rho) \right] \sin(2\omega t) + Y_0(k_0\rho) J_1(k_0\rho) \cos^2\omega t - J_0(k_0\rho) Y_1(k_0\rho) \sin^2\omega t \right\} \hat{\rho}; \quad \rho \geq R$$

$$\langle \vec{S}(\rho,t) \rangle = \frac{1}{32} Z_0 (I_0 k_0)^2 J_0^2(k_0 R) \Big[\overbrace{Y_0(k_0\rho) J_1(k_0\rho) - J_0(k_0\rho) Y_1(k_0\rho)}^{= 2/\pi k_0 \rho \text{ ; see the note on Bessel functions.}} \Big] \hat{\rho} \Rightarrow$$

$$\langle \vec{S}(\rho,t) \rangle = \frac{Z_0}{16\pi k_0 \rho} \left[I_0 k_0 J_0(k_0 R) \right]^2 \hat{\rho} = \frac{Z_0 I_0^2 k_0}{16\pi\rho} J_0^2(k_0 R) \hat{\rho}; \quad \rho \geq R$$

The radiated power that leaves a cylinder of radius ρ and unit length is thus found to be:

$$\text{Total radiated power per unit length} = 2\pi\rho \langle S \rangle = \frac{1}{8} Z_0 I_0^2 k_0 J_0^2(k_0 R)$$

Now, the $\vec{E}$-field at the cylinder surface is obtained by letting $\rho = R$ in the expression for the $\vec{E}$-field:

$$\vec{E}(\vec{r},t) = \frac{1}{4} Z_0 I_0 k_0 J_0(k_0 R) \left[Y_0(k_0 R) \cos\omega t - J_0(k_0 R) \sin\omega t \right] \hat{z}$$

This $\vec{E}$-field opposes the $\vec{E}$-field of the external source that drives the current within the walls of the cylindrical conductor. In a perfect conductor, the net $\vec{E}$-field must be zero; therefore, the external field is exactly equal to the negative of the above, namely,

$$\vec{E}_{external} = -\frac{1}{4} Z_0 I_0 k_0 J_0(k_0 R) \left[Y_0(k_0 R) \cos\omega t - J_0(k_0 R) \sin\omega t \right] \hat{z}$$

The electric power per unit length of the cylinder supplied by the external source is:

$$\text{Electrical power per unit length} = E_{external} I = E_{external} I_0 \sin\omega t =$$

$$-\frac{1}{4} Z_0 I_0^2 k_0 J_0(k_0 R) \left[Y_0(k_0 R) \cos\omega t \sin\omega t - J_0(k_0 R) \sin^2\omega t \right]$$

$$= \frac{1}{8} Z_0 I_0^2 k_0 J_0(k_0 R) \left[J_0(k_0 R) - J_0(k_0 R) \cos(2\omega t) + Y_0(k_0 R) \sin(2\omega t) \right]$$

Time averaged electrical power per unit length $= \frac{1}{8} Z_0 I_0^2 k_0 J_0^2(k_0 R)$ is thus in complete agreement with the expression obtained earlier for the time-averaged radiated power per unit length of the cylinder.

d) Infinite sheet of current in the yz-plane; $\vec{J}_s(t) = J_{s0} \sin(\omega t)\hat{z}$; HW#7, Prob. 3:

$$\begin{cases} \vec{E}(\vec{r},t) = \frac{1}{2} Z_0 J_{s0} \sin(k_0 x - \omega t)\hat{z}; & x \geq 0 \\ \vec{H}(\vec{r},t) = -\frac{1}{2} J_{s0} \sin(k_0 x - \omega t)\hat{y}; & x \geq 0 \end{cases}$$

$$\vec{S}(\vec{r},t) = \vec{E} \times \vec{H} = \frac{1}{4} Z_0 J_{s0}^2 \sin^2(k_0 x - \omega t)\hat{x}; \quad x \geq 0$$

$$\langle \vec{S}(\vec{r},t) \rangle = \frac{1}{8} Z_0 J_{s0}^2 \hat{x}; \quad x \geq 0$$ ← Radiated power per unit area ∥ to YZ-plane.

$\vec{E}$-field at the surface of the conductor is obtained by setting $x=0$ in the expression for the $\vec{E}$-field:

$\vec{E}(x=0,t) = -\frac{1}{2} Z_0 J_{s0} \sin(\omega t)\hat{z}$ (in the YZ-plane)

$$\vec{E}_{external} = -\vec{E}(x=0,t) = \frac{1}{2} Z_0 J_{s0} \sin(\omega t)\hat{z}$$

Electrical power supplied to the sheet (per unit area) $= \vec{E}_{external} \cdot \vec{J}_s =$

$$\left[\frac{1}{2} Z_0 J_{s0} \sin(\omega t)\right] J_{s0} \sin(\omega t) = \frac{1}{2} Z_0 J_{s0}^2 \sin^2(\omega t)$$

The time-averaged electrical power (per unit area of the sheet) is thus $\frac{1}{4} Z_0 J_{s0}^2$, which is twice the radiated power in the $x>0$ region. Considering that the same amount of optical power is radiated in the $x<0$ region, there is complete agreement between the total electrical power supplied and the total optical power radiated.

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e) Infinite sheet of current $\vec{J}_s(\vec{r},t) = J_{s0}\sin(\omega t - \kappa z)\hat{z}$ located in yz-plane at x=0.

Case I: $\kappa < k_0$) From HW#7 Prob. 4 and also from the discussion in the class we have:

$$\left\{\begin{array}{l} \vec{E}(\vec{r},t) = -\frac{1}{2} Z_0 J_{s0}\left(\frac{\kappa}{k_0}\hat{x} - \sqrt{1-(\kappa/k_0)^2}\,\hat{z}\right)\sin\left(\sqrt{k_0^2-\kappa^2}\,x + \kappa z - \omega t\right); \quad x \geq 0 \\ H(\vec{r},t) = -\frac{1}{2} J_{s0}\sin\left(\sqrt{k_0^2-\kappa^2}\,x + \kappa z - \omega t\right)\hat{y}; \quad x \geq 0 \end{array}\right.$$

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \frac{1}{4} Z_0 J_{s0}^2\left(\frac{\kappa}{k_0}\hat{z} + \sqrt{1-(\kappa/k_0)^2}\,\hat{x}\right)\sin^2\left[k_0\left(\sqrt{1-(\kappa/k_0)^2}\,x + \frac{\kappa}{k_0}z - ct\right)\right]$$

We define a unit vector $\vec{\sigma} = \sqrt{1-(\kappa/k_0)^2}\,\hat{x} + (\kappa/k_0)\hat{z}$ to represent the direction of propagation of energy. We'll have:

$$\vec{S}(\vec{r},t) = \frac{1}{4} Z_0 J_{s0}^2 \sin^2[k_0(\vec{\sigma}\cdot\vec{r} - ct)]\vec{\sigma}; \quad x \geq 0$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{8} Z_0 J_{s0}^2\vec{\sigma}; \quad x \geq 0$$

$\vec{E}$-field acting on the current sheet $= \vec{E}(x=0,y,z,t) = -\frac{1}{2} Z_0 J_{s0}\left(\frac{\kappa}{k_0}\hat{x} - \sqrt{1-(\kappa/k_0)^2}\,\hat{z}\right) \times \sin(\kappa z - \omega t)$

The only component of $\vec{E}$ that does work on the current is E_z; the external field applied by the power supply to counter this component of the radiated field is, therefore, given by:

$$\vec{E}_{external} = -E_z(x=0,y,z,t)\hat{z} = +\frac{1}{2} Z_0 J_{s0}\sqrt{1-(\kappa/k_0)^2}\sin(\omega t - \kappa z)\hat{z}$$

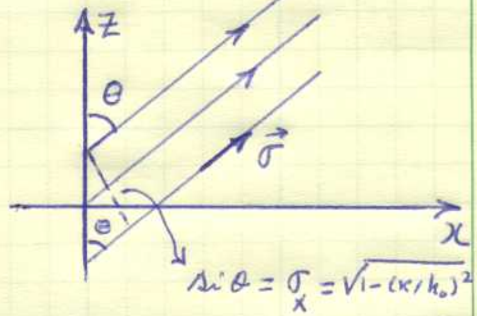
Electrical power per unit area supplied by the external source =

$$\vec{E}_{external}\cdot\vec{J}_s = \frac{1}{2} Z_0 J_{s0}^2\sqrt{1-(\kappa/k_0)^2}\sin^2(\omega t - \kappa z)$$

Time-averaged electrical power per unit area of the sheet $= \frac{1}{4} Z_0 J_{s0}^2\sqrt{1-(\kappa/k_0)^2}$

Again, the factor of 2 difference between electrical and optical powers is due to the fact that the sheet radiates into both $x>0$ and $x<0$ regions.

The factor $\sqrt{1-(\kappa/k_0)^2}$ is the obliquity factor, accounting for the reduced cross-section of the beam when projected onto the plane perpendicular to the propagation direction $\vec{\sigma}$.



$\sin\theta = \sigma_x = \sqrt{1-(\kappa/k_0)^2}$

obliquity factor

Case II: $\kappa > k_0$)

$$\begin{cases} \vec{E}(\vec{r},t) = -\frac{1}{2} Z_0 J_{s0}\, e^{-\sqrt{\kappa^2-k_0^2}\,x} \left[\frac{\kappa}{k_0}\sin(\kappa z-\omega t)\hat{x} - \sqrt{(\kappa/k_0)^2-1}\,\cos(\kappa z-\omega t)\hat{z}\right]; \quad x \geq 0 \\ H(\vec{r},t) = -\frac{1}{2} J_{s0}\, e^{-\sqrt{\kappa^2-k_0^2}\,x} \sin(\kappa z-\omega t)\hat{y}; \quad x \geq 0 \end{cases}$$

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \frac{1}{4} Z_0 J_{s0}^2\, e^{-2\sqrt{\kappa^2-k_0^2}\,x} \left\{\frac{\kappa}{k_0}\sin^2(\kappa z-\omega t)\hat{z} + \frac{1}{2}\sqrt{(\kappa/k_0)^2-1}\,\sin[2(\kappa z-\omega t)]\hat{x}\right\}$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\kappa Z_0 J_{s0}^2}{8k_0} \exp\left(-2\sqrt{\kappa^2-k_0^2}\,x\right)\hat{z}$$

Note that in this case, when $\kappa > k_0$, the optical energy does not leave the vicinity of the current sheet; rather it lingers within a certain range of the x-axis, up to a distance of $\sim \lambda_0/\sqrt{(\kappa/k_0)^2-1}$, and propagates along the z-axis.

E-field acting on the current sheet $= -E_z(x=0,y,z,t) = -\frac{1}{2} Z_0 J_{s0}\sqrt{(\kappa/k_0)^2-1}\,\cos(\kappa z-\omega t)$
$\quad$ ext

Electrical power (per unit area) delivered to the sheet $= \vec{E}_{\text{external}} \cdot \vec{J}_s$

$$= \frac{1}{4} Z_0 J_{s0}^2 \sqrt{(\kappa/k_0)^2-1}\,\sin[2(\kappa z-\omega t)]$$

$\Rightarrow$ Time-averaged electrical power per unit area $= 0$.

This shows that the optical energy in the system is recycled, i.e., what goes out at $z=+\infty$ returns at $z=-\infty$. Therefore, since no energy leaves the system in the steady-state, there is no need for external energy to be supplied.

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f) Infinite sheet of current $\vec{J}_s(\vec{r},t) = J_{so}\sin(\omega t - \kappa y)\hat{z}$ located in the yz-plane at $x=0$.

Case I: $\kappa < k_0$) From HW #7, problem 5, we have:

$$\begin{cases} \vec{E}(\vec{r},t) = -\dfrac{Z_0 J_{so}}{2\sqrt{1-(\kappa/k_0)^2}}\sin\left(\omega t - \kappa y - \sqrt{k_0^2-\kappa^2}\,x\right)\hat{z}; & x \geq 0 \\ H(\vec{r},t) = -\dfrac{J_{so}}{2\sqrt{1-(\kappa/k_0)^2}}\left(\dfrac{\kappa}{k_0}\hat{x} - \sqrt{1-(\kappa/k_0)^2}\,\hat{y}\right)\sin\left(\omega t - \kappa y - \sqrt{k_0^2-\kappa^2}\,x\right); & x \geq 0 \end{cases}$$

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \frac{Z_0 J_{so}^2}{4[1-(\kappa/k_0)^2]}\left(\frac{\kappa}{k_0}\hat{y} + \sqrt{1-(\kappa/k_0)^2}\,\hat{x}\right)\sin^2\left[k_0\left(\frac{\kappa}{k_0}y + \sqrt{1-(\kappa/k_0)^2}\,x - ct\right)\right]$$

Defining the unit vector $\vec{\sigma} = \sqrt{1-(\kappa/k_0)^2}\,\hat{x} + (\kappa/k_0)\hat{y}$ to represent the direction of propagation of energy, we write:

$$\vec{S}(\vec{r},t) = \frac{Z_0 J_{so}^2}{4[1-(\kappa/k_0)^2]}\sin^2[k_0(\vec{\sigma}\cdot\vec{r} - ct)]\,\vec{\sigma}; \qquad x \geq 0$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{Z_0 J_{so}^2}{8[1-(\kappa/k_0)^2]}\vec{\sigma}; \qquad x \geq 0$$

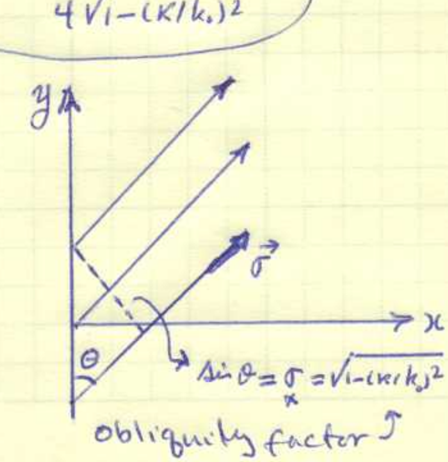
$$\vec{E}_{external} \text{ acting on the sheet} = -E_z(x=0,y,z,t)\hat{z} = \frac{Z_0 J_{so}}{2\sqrt{1-(\kappa/k_0)^2}}\sin(\omega t - \kappa y)\hat{z}$$

External electrical power per unit area delivered to the sheet by the external source =

$$\vec{E}_{external}\cdot\vec{J}_s = \frac{Z_0 J_{so}^2}{2\sqrt{1-(\kappa/k_0)^2}}\sin^2(\omega t - \kappa z)$$

$$\text{Time-averaged electrical power per unit area} = \frac{Z_0 J_{so}^2}{4\sqrt{1-(\kappa/k_0)^2}}$$

As before, the factor of 2 difference between optical and electrical powers can be accounted for by the radiation of the sheet into both $x \geq 0$ and $x \leq 0$ regions, and the factor $\sqrt{1-(\kappa/k_0)^2}$ is the obliquity factor.



Case II: $\kappa > k_0$)

$$\begin{cases} \vec{E}(\vec{r},t) = -\dfrac{Z_0 J_{s0}}{2\sqrt{(\kappa/k_0)^2-1}} \exp(-\sqrt{\kappa^2-k_0^2}\,x)\cos(\omega t-\kappa y)\hat{z}; & x \geq 0 \\ H(\vec{r},t) = -\dfrac{J_{s0}}{2\sqrt{(\kappa/k_0)^2-1}}\left[\dfrac{\kappa}{k_0}\cos(\omega t-\kappa y)\hat{x} - \sqrt{(\kappa/k_0)^2-1}\,\sin(\omega t-\kappa y)\hat{y}\right]e^{-\sqrt{\kappa^2-k_0^2}\,x}; & x \geq 0 \end{cases}$$

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \frac{Z_0 J_{s0}^2}{4[(\kappa/k_0)^2-1]}\exp\left(-2\sqrt{\kappa^2-k_0^2}\,x\right)\left\{\frac{\kappa}{k_0}\cos^2(\omega t-\kappa y)\hat{y} + \tfrac{1}{2}\sqrt{(\kappa/k_0)^2-1}\,\sin[2(\omega t-\kappa y)]\hat{x}\right\}$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{(\kappa/k_0)Z_0 J_{s0}^2}{8[(\kappa/k_0)^2-1]}\exp\left(-2k_0\sqrt{(\kappa/k_0)^2-1}\,x\right)\hat{y}; \quad x \geq 0$$

$$\vec{E}_{\text{external}} = -E_z(x=0,y,z,t)\hat{z} = \frac{Z_0 J_{s0}}{2\sqrt{(\kappa/k_0)^2-1}}\cos(\omega t-\kappa y)\hat{z}$$

Electrical power per unit area supplied to the current sheet by the external source $= \vec{E}_{\text{external}} \cdot \vec{J}_s = \dfrac{Z_0 J_{s0}^2}{4\sqrt{(\kappa/k_0)^2-1}}\sin[2(\omega t-\kappa y)]$

Time-averaged electrical power per unit area $= 0$.

As before, the optical energy in this case is recycled; what goes out at $y=\infty$ returns from $y=-\infty$. Since no energy leaves the system in the steady-state, there is no need for external energy to be supplied. All the optical energy in the system is confined to the vicinity of the current sheet, within a distance of $\sim \lambda_0/\sqrt{(\kappa/k_0)^2-1}$ from the YZ-plane. The optical energy does propagate, however, parallel to the current sheet and in the $\hat{y}$-direction.