Problem 36)

a) $\vec{\nabla}\times\vec{H}=\vec{J} \Rightarrow \oint \vec{H}\cdot d\vec{\ell} = \int \vec{J}\cdot d\vec{s} \Rightarrow 2\pi\rho H_\phi = \begin{cases} (\rho^2/R^2) I_0 & \rho \le R \\ I_0 & \rho \ge R \end{cases}$

$$\Rightarrow \vec{H}(\rho,\phi,z) = \begin{cases} \dfrac{\rho I_0}{2\pi R^2}\,\hat{\phi} & \rho \le R \\ [I_0/(2\pi\rho)]\,\hat{\phi} & \rho \ge R \end{cases}$$

b) Inside the rod, the $\vec{E}$-field is uniform ($\vec{E}=E_0\hat{z}$). Therefore, its curl and divergence must be zero. Outside the rod, we have

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)\hat{\phi} = E_0\left\{\frac{-1}{\rho \ln(R_0/R)} + \frac{1/\rho}{\ln(R_0/R)}\right\}\hat{\phi} = 0.$$

Also note that at $\rho = R$ we have $E_z(R,\phi,z) = E_0$ on both sides of the cylinder surface; in other words, $E_z$ is continuous at the surface of the cylindrical rod. The fact that $\vec{\nabla}\times\vec{E}=0$ throughout the entire space ($0\le z \le L$, $0 \le \rho \le R_0$) indicates that a potential function $\psi(\rho,\phi,z)$ can be specified (within the relevant space) in this static problem, as it should. Finally, $\vec{\nabla}\cdot\vec{E}$ outside the rod is given by

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{\partial E_z}{\partial z} = E_0\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{L-z}{\ln(R_0/R)}\right] + \frac{\partial}{\partial z}\left[1 - \frac{\ln(\rho/R)}{\ln(R_0/R)}\right]\right\} = 0 \checkmark$$

c) The $\vec{E}$-field perpendicular to the cylinder surface, $E_\rho$, is discontinuous. Therefore, $\sigma_s(\rho=R,\phi,z) = \epsilon_0\left[E_\rho(\rho=R^+,\phi,z) - E_\rho(\rho=R^-,\phi,z)\right] = \epsilon_0 E_0 \dfrac{L-z}{R\ln(R_0/R)}$

Note that the charge density is strong at $z=0$, decreasing linearly with $z$, until it drops to zero at $z=L$. To get a feel for the numbers involved, suppose $E_0 = 1000$ V/m, $L = 1$ km, $R = 1$ mm, $R_0 = 1$ m, $z=0$. We find (using $\epsilon_0 = 8.854\times10^{-12}$ F/m) that $\sigma_s(\rho=R,\phi,z=0) = 1.3\times10^{-3}$ Coulomb/m$^2$.

Compare this with the charge density at the cylinder surface if only one atomic layer had been left "naked," i.e., without electrons. A single atom typically is about $1\,\mathring{A} = 10^{-10}\,m$ in diameter; therefore, a square meter of the surface area contains about $10^{20}$ atoms. Multiply this with the electronic charge of $e = 1.6\times10^{-19}\,C$, and you get $\sigma_s \cong 16\ C/m^2$; which is 4 orders of magnitude greater than $\sigma_s$ in the present problem.

The surface charge distribution on the cylindrical rod is responsible for creating the field $\vec{E} = E_0\hat{z}$ inside the conductor; The linear decrease of $\sigma_s$ with $z$ allows for the uniform $\vec{E}$-field inside the rod to be directed along the $z$-axis. Note that the voltage drop along the length of the conductor is $E_0 L$, which is equal to the initial voltage at $z=0$, given by $V_0 = \int_{\rho=R}^{R_0} E_\rho(\rho,\phi,z=0)\,d\rho = \frac{E_0 L}{\ln(R_0/R)}\int_R^{R_0}\frac{d\rho}{\rho} = E_0 L$ ✓

d) $\vec{S} = \vec{E}\times\vec{H}$

outside the rod $\vec{S} = E_0\left\{\frac{L-z}{\rho\ln(R_0/R)}\hat{\rho} + \left[1-\frac{\ln(\rho/R)}{\ln(R_0/R)}\right]\hat{z}\right\}\times\frac{I_0}{2\pi\rho}\hat{\phi}$

$$\Rightarrow \vec{S}(\rho,\phi,z) = E_0 I_0\frac{L-z}{2\pi\rho^2\ln(R_0/R)}\hat{z} - \frac{E_0 I_0}{2\pi\rho}\left[1-\frac{\ln(\rho/R)}{\ln(R_0/R)}\right]\hat{\rho}$$

The energy enters the system at $z=0$, through the $z$-component of the Poynting vector, $S_z(\rho,\phi,z=0) = \frac{E_0 I_0 L}{2\pi\rho^2\ln(R_0/R)}$ :

$$\int_{\rho=R}^{R_0}\frac{E_0 I_0 L}{2\pi\rho^2\ln(R_0/R)}\,2\pi\rho\,d\rho = E_0 I_0 L = V_0 I_0 \ ✓$$

No energy enters the system along the radial direction, because at $\rho = R_0$ the radial component of the $\vec{S}$-field is zero:

$$S_\rho(\rho=R_0,\phi,z) = -\frac{E_0 I_0}{2\pi\rho}\left[1-\frac{\ln(R_0/R)}{\ln(R_0/R)}\right] = 0\ ✓$$

However, at other points along the $\rho$-direction, $S_\rho$ is non-zero and points toward the rod. The integral of $S_\rho$ around a cylinder of radius $\rho$ and unit length is thus given by:

$$\int_{\phi=0}^{2\pi}\int_{z=z_0}^{z_0+1} S_\rho(\rho,\phi,z)\,\rho\,d\phi\,dz = -E_0 I_0\left[1-\frac{\ln(\rho/R)}{\ln(R_0/R)}\right].$$

The energy entering the system along the $z$-axis is thus seen to turn toward the rod (i.e., in the $\rho$-direction) at a rate that is independent of $z$, but increases as $\rho$ moves from $R_0$ to $R$. At $\rho = R$, i.e., at the cylinder surface, the energy enters the cylinder at a rate of $E_0 I_0$ per unit length per second. By the time it gets to $z = L$, all the energy has gone into the rod (and converted into heat); that is why $S_z(\rho,\phi,z=L) = 0$, i.e., no energy leaves the system at $z = L$ along the $z$-axis.

Inside the rod $\vec{S} = E_0\hat{z} \times \frac{\rho I_0}{2\pi R^2}\hat{\phi} = -\frac{E_0 I_0 \rho}{2\pi R^2}\hat{\rho}$

$$\int_{\phi=0}^{2\pi}\int_{z=z_0}^{z_0+1} S_\rho(\rho,\phi,z)\,\rho\,d\phi\,dz = -E_0 I_0(\rho/R)^2$$

The energy thus enters the rod radially at the surface, at a rate of $E_0 I_0$ per unit length per second. It is deposited (in the form of heat) as it moves toward the center, until it reaches zero at $\rho = 0$. Note that no energy enters from the bottom of the rod at $z = 0$; all energy flows in radially from the air outside the rod.