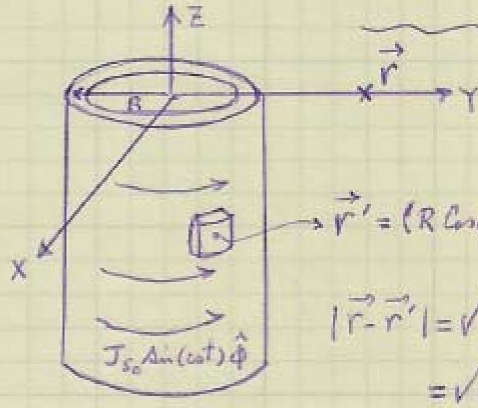
**Solutions** **Opti 501**

Problem 35)

a) $\vec{\nabla}\cdot\vec{J}_s + \frac{\partial \sigma_s}{\partial t} = 0 \Rightarrow \frac{\partial \sigma_s}{\partial t} = -\vec{\nabla}\cdot\vec{J}_s = -\frac{1}{R}\frac{\partial}{\partial \phi} J_{s\phi} = 0 \Rightarrow \sigma_s(R,\phi,z,t) = 0$

b)




observation point $\vec{r} = (0, y, 0)$

$\vec{r}\,' = (R\cos\phi', R\sin\phi', z')$

$$|\vec{r}-\vec{r}\,'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$
$$= \sqrt{R^2\cos^2\phi' + (y - R\sin\phi')^2 + z'^2}$$
$$= \sqrt{R^2 + y^2 + z'^2 - 2Ry\sin\phi'}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\int_{z'=-\infty}^{\infty} \frac{\vec{J}_s(R,\phi',z',t-|\vec{r}-\vec{r}\,'|/c)}{|\vec{r}-\vec{r}\,'|} R\,d\phi'\,dz' \Rightarrow$$

Use symmetry to replace $y$ with $\rho$. The only component of $\vec{A}$ will be in the $\hat{\phi}$ direction.

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s_0} R\hat{\phi}}{\pi}\int_{\phi'=-\pi/2}^{\pi/2}\sin\phi'\int_{z'=0}^{\infty} \frac{\sin\left[\omega t - k_0\sqrt{R^2+\rho^2+z'^2-2R\rho\sin\phi'}\right]}{\sqrt{R^2+\rho^2+z'^2-2R\rho\sin\phi'}}\,d\phi'\,dz'$$

In the above equation $\sin\phi'$ is introduced to project $\vec{J}_s$ at $\phi'$ onto $\vec{A}$ in the $\hat{\phi}$-direction. Since the observation point is located on the $y$-axis (for reasons of symmetry), the contributions of $\vec{J}_s$ along the $y$-axis cancel out, leaving only the contributions $\perp$ to $y$-axis, which is the $\hat{\phi}$-direction.

$$\vec{A}(\vec{r},t) = -\frac{1}{2}\mu_0 J_{s0} R\hat{\phi}\left\{\left[\int_{\phi'=-\pi/2}^{\pi/2} \sin\phi'\, Y_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\sin\phi'}\right) d\phi'\right]\sin\omega t\right.$$

$$\left. + \left[\int_{\phi'=-\pi/2}^{\pi/2} \sin\phi'\, J_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\sin\phi'}\right) d\phi'\right]\cos\omega t\right\}$$

The integrals in the above equation can be transformed into the integrals listed in Gradshteyn + Ryzhik, page 741, #6.684 (1,2) after some manipulation:

$$\int_{\phi'=-\pi/2}^{\pi/2} \sin\phi'\, Y_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\sin\phi'}\right) d\phi' \underset{\theta=\frac{\pi}{2}-\phi'}{=} \int_{\theta=0}^{\pi} \cos\theta\, Y_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\cos\theta}\right) d\theta$$

Integration by parts

$$= \sin\theta\, Y_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\cos\theta}\right)\Big|_0^{\pi} + \int_{\theta=0}^{\pi} \sin\theta\, \frac{k_0 R\rho\sin\theta}{\sqrt{\cdots}}\, Y_1\!\left(k_0\sqrt{\cdots}\right) d\theta$$

$$= k_0 R\rho \int_{\theta=0}^{\pi} \sin^2\theta\, \frac{Y_1\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\cos\theta}\right)}{\sqrt{R^2+\rho^2-2R\rho\cos\theta}}\, d\theta = \begin{cases} \pi Y_1(k_0R)J_1(k_0\rho); & \rho\le R \\ \pi J_1(k_0R)Y_1(k_0\rho); & \rho\ge R \end{cases}$$

Similarly, $\displaystyle\int_{\phi'=-\pi/2}^{\pi/2} \sin\phi'\, J_0\!\left(k_0\sqrt{R^2+\rho^2-2R\rho\sin\phi'}\right) d\phi' = \pi J_1(k_0R)J_1(k_0\rho)$.

Consequently,

$$\vec{A}(\vec{r},t) = -\frac{\pi}{2}\mu_0 J_{s0} R\hat{\phi}\begin{cases} \left[Y_1(k_0R)\sin\omega t + J_1(k_0R)\cos\omega t\right]J_1(k_0\rho); & \rho\le R \\ J_1(k_0R)\left[Y_1(k_0\rho)\sin\omega t + J_1(k_0\rho)\cos\omega t\right]; & \rho\ge R \end{cases}$$

c) $\displaystyle\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{\hat{z}}{\mu_0\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)$

But $\displaystyle\frac{d}{d\rho}\left[\rho J_1(k_0\rho)\right] = J_1(k_0\rho) + k_0\rho J_1'(k_0\rho) = J_1(k_0\rho) + k_0\rho\left[J_0(k_0\rho) - \frac{1}{k_0\rho}J_1(k_0\rho)\right]$

$= k_0\rho J_0(k_0\rho)$. Similarly, $\displaystyle\frac{d}{d\rho}\left[\rho Y_1(k_0\rho)\right] = k_0\rho Y_0(k_0\rho)$. Therefore,

$$\vec{H}(\vec{r},t) = -\frac{\pi}{2} J_{s0} k_0 R \hat{z} \begin{cases} [Y_1(k_0R)\sin\omega t + J_1(k_0R)\cos\omega t]\, J_0(k_0\rho); & \rho \le R \\ J_1(k_0R)\,[Y_0(k_0\rho)\sin\omega t + J_0(k_0\rho)\cos\omega t]; & \rho > R \end{cases}$$

Dis Continuity of $\vec{H}$ at the Cylinder surface $= \vec{H}(\text{out}) - \vec{H}(\text{in}) =$

$$-\frac{\pi}{2} J_{s0} k_0 R \hat{z} \left[Y_0(k_0R) J_1(k_0R) - J_0(k_0R) Y_1(k_0R)\right] \sin\omega t = -\frac{\pi}{2} J_{s0} k_0 R \hat{z} \frac{2}{\pi k_0 R} \sin\omega t$$

$= -J_{s0} \sin\omega t\, \hat{z}$ ← Equal to surface current density and $\perp$ to its direction

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\pi}{2} \mu_0 \omega J_{s0} R \hat{\phi} \begin{cases} [Y_1(k_0R)\cos\omega t - J_1(k_0R)\sin\omega t]\, J_1(k_0\rho); & \rho \le R \\ J_1(k_0R)\,[Y_1(k_0\rho)\cos\omega t - J_1(k_0\rho)\sin\omega t]; & \rho \ge R \end{cases} \Rightarrow$$

$$\vec{E}(\vec{r},t) = \frac{\pi}{2} Z_0 J_{s0} k_0 R \hat{\phi} \begin{cases} [Y_1(k_0R)\cos\omega t - J_1(k_0R)\sin\omega t]\, J_1(k_0\rho); & \rho \le R \\ J_1(k_0R)\,[Y_1(k_0\rho)\cos\omega t - J_1(k_0\rho)\sin\omega t]; & \rho \ge R \end{cases}$$

d)

Inside the Cylinder: $\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = -\frac{\pi^2}{4} Z_0 J_{s0}^2 k_0^2 R^2 \hat{\rho} \Big[Y_1^2(k_0R)\sin\omega t\cos\omega t +$

$J_1(k_0R) Y_1(k_0R) \cos^2\omega t - J_1(k_0R) Y_1(k_0R) \sin^2\omega t - J_1^2(k_0R)\sin\omega t\cos\omega t\Big] J_0(k_0\rho) J_1(k_0\rho)$

$$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = -\frac{\pi^2}{4} Z_0 J_{s0}^2 k_0^2 R^2 \hat{\rho} \left[\frac{1}{2} J_1(k_0R) Y_1(k_0R) - \frac{1}{2} J_1(k_0R) Y_1(k_0R)\right] J_0(k_0\rho) J_1(k_0\rho) = 0$$

Outside the Cylinder: $\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = -\frac{\pi^2}{4} Z_0 J_{s0}^2 k_0^2 R^2 \hat{\rho}\, J_1^2(k_0R) \times$

$\Big[Y_0(k_0\rho) Y_1(k_0\rho)\sin\omega t\cos\omega t - J_0(k_0\rho) J_1(k_0\rho)\sin\omega t\cos\omega t - Y_0(k_0\rho) J_1(k_0\rho)\sin^2\omega t$

$+ J_0(k_0\rho) Y_1(k_0\rho)\cos^2\omega t\Big] \Rightarrow$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\pi^2}{4} Z_0 J_{s0}^2 k_0^2 R^2 J_1^2(k_0R)\left[\frac{1}{2} Y_0(k_0\rho) J_1(k_0\rho) - \frac{1}{2} J_0(k_0\rho) Y_1(k_0\rho)\right]\hat{\rho}$$

$$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{\pi R^2}{4\rho} Z_0 J_{s0}^2 k_0 J_1^2(k_0R)\, \hat{\rho}; \quad \rho > R$$

Note that the radiated power is inversely proportional to $\rho$ in this Cylindrical geometry.