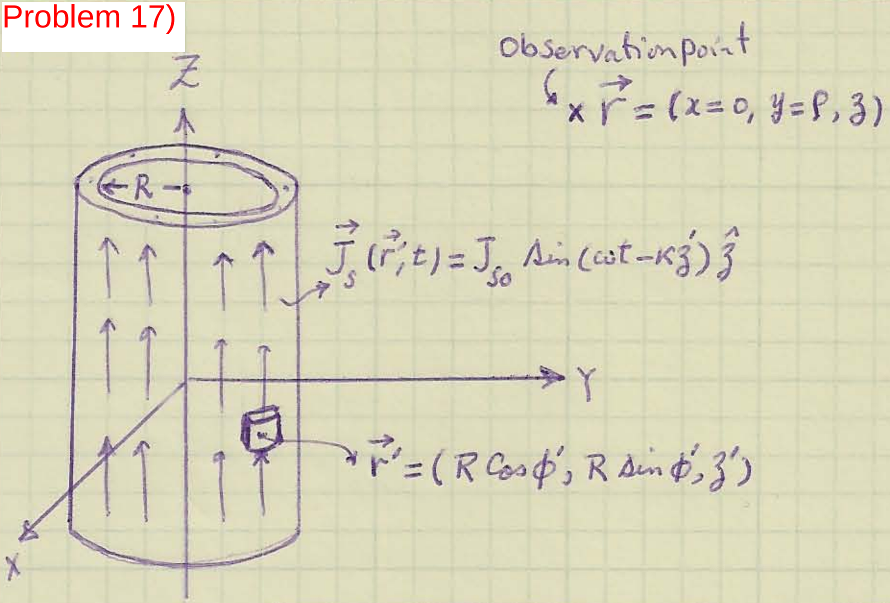
Problem 17)



$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \sqrt{R^2\cos^2\phi' + (\rho - R\sin\phi')^2 + (z-z')^2}$$

$$= \sqrt{R^2+\rho^2-2R\rho\sin\phi' + (z-z')^2}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{z'=-\infty}^{\infty}\int_{\phi'=0}^{2\pi} \frac{J_{s0}\sin(\omega t - k_0|\vec{r}-\vec{r}'| - \kappa z')\hat{z}}{|\vec{r}-\vec{r}'|} R\,d\phi'\,dz'$$

Change of variable $u = z'-z$

$$= \frac{\mu_0 J_{s0} R\hat{z}}{4\pi}\int_{u=-\infty}^{\infty}\int_{\phi'=0}^{2\pi} \frac{\sin(\omega t - k_0\sqrt{R^2+\rho^2-2R\rho\sin\phi'+u^2} - \kappa z - \kappa u)}{\sqrt{R^2+\rho^2-2R\rho\sin\phi'+u^2}}\,d\phi'\,du$$

$$= \frac{\mu_0 J_{s0} R\hat{z}}{2\pi}\int_{\phi'=-\pi/2}^{\pi/2}\int_{u=-\infty}^{\infty} \frac{1}{\sqrt{\cdots}}\left\{\sin(\omega t-\kappa z)\cos(k_0\sqrt{\cdots}+\kappa u) - \cos(\omega t - \kappa z)\sin(k_0\sqrt{\cdots}+\kappa u)\right\} du\,d\phi'$$

$$= \frac{\mu_0 J_{s0} R\hat{z}}{2\pi}\int_{\phi'=-\pi/2}^{\pi/2}\int_{u=-\infty}^{\infty} \frac{1}{\sqrt{\cdots}}\Big\{\sin(\omega t-\kappa z)\left[\cos(k_0\sqrt{\cdots})\cos(\kappa u) - \sin(k_0\sqrt{\cdots})\sin(\kappa u)\right]$$
$$- \cos(\omega t-\kappa z)\left[\sin(k_0\sqrt{\cdots})\cos(\kappa u) + \cos(k_0\sqrt{\cdots})\sin(\kappa u)\right]\Big\} du\,d\phi'$$

The terms containing $\sin(\kappa u)$ are odd functions of u and integrate to zero. Thus,

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s0} R\hat{z}}{\pi}\int_{\phi'=-\pi/2}^{\pi/2} d\phi'\left\{\left[\int_{u=0}^{\infty}\frac{\cos(k_0\sqrt{\cdots})}{\sqrt{\cdots}}\cos(\kappa u)\,du\right]\sin(\omega t - \kappa z)\right.$$
$$\left. - \left[\int_{u=0}^{\infty}\frac{\sin(k_0\sqrt{\cdots})}{\sqrt{\cdots}}\cos(\kappa u)\,du\right]\cos(\omega t-\kappa z)\right\}$$

From Gradshteyn + Ryzhik, page 472, # 3.876, 1,2, we have:

$$\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}} \cos bx\,dx = \begin{cases} \frac{\pi}{2} J_0(a\sqrt{p^2-b^2}) & 0<b<p \\ 0 & b>p>0 \end{cases}$$

$$\int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}} \cos bx\,dx = \begin{cases} -\frac{\pi}{2} Y_0(a\sqrt{p^2-b^2}) & 0<b<p \\ K_0(a\sqrt{b^2-p^2}) & b>p>0 \end{cases}$$

We may thus write the vector potential function as follows (for $k_0>\kappa$):

$$\vec{A}(\vec{r},t) = -\frac{1}{2}\mu_0 J_{s0} R\hat{z} \left\{ \left[\int_{\phi'=-\pi/2}^{\pi/2} Y_0\left(\sqrt{(k_0^2-\kappa^2)(R^2+\rho^2-2R\rho\sin\phi')}\right) d\phi' \right] \sin(\omega t-\kappa z) + \left[\int_{\phi'=-\pi/2}^{\pi/2} J_0\left(\sqrt{(k_0^2-\kappa^2)(R^2+\rho^2-2R\rho\sin\phi')}\right) d\phi' \right] \cos(\omega t-\kappa z) \right\}$$

And, for $k_0<\kappa$, we'll have:

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s0} R\hat{z}}{\pi} \left[\int_{\phi'=-\pi/2}^{\pi/2} K_0\left(\sqrt{(\kappa^2-k_0^2)(R^2+\rho^2-2R\rho\sin\phi')}\right) d\phi' \right] \sin(\omega t-\kappa z).$$

From Gradshteyn + Ryzhik, page 741, # 6.684 (1,2), we have (for $k_0>\kappa$):

$$\vec{A}(\vec{r},t) = -\frac{1}{2}\mu_0 \pi J_{s0} R\hat{z} \begin{cases} \left[Y_0(\sqrt{k_0^2-\kappa^2}\,R)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}\,R)\cos(\omega t-\kappa z)\right] J_0(\sqrt{k_0^2-\kappa^2}\,\rho) & (\rho<R) \text{ inside cylinder,} \\ J_0(\sqrt{k_0^2-\kappa^2}\,R)\left[Y_0(\sqrt{k_0^2-\kappa^2}\,\rho)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}\,\rho)\cos(\omega t-\kappa z)\right] & (\rho>R) \text{ outside cylinder.} \end{cases}$$

As for the case of $k_0<\kappa$, we find in Grad. + Ryzh., page 772, Eq. (6.794) #1:

$$K_0(\sqrt{a^2+b^2-2ab\sin\phi'}) = \frac{2}{\pi}\int_0^\infty K_{ix}(a)\,K_{ix}(b)\,\cosh[(\tfrac{\pi}{2}+\phi')x]\,dx \qquad \text{(after substituting } \tfrac{\pi}{2}-\phi' \text{ for } \phi)$$

Thus $\displaystyle\int_{-\pi/2}^{\pi/2} K_0(\sqrt{a^2+b^2-2ab\sin\phi'})\,d\phi' = \frac{2}{\pi}\int_0^\infty K_{ix}(a)K_{ix}(b)\int_{-\pi/2}^{\pi/2}\cosh[(\tfrac{\pi}{2}+\phi')x]\,d\phi'\,dx = \frac{2}{\pi}\int_0^\infty K_{ix}(a)K_{ix}(b)\frac{1}{x}\int_0^{\pi x}\cosh\varphi\,d\varphi\,dx$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} K_0(\sqrt{a^2+b^2-2ab\sin\varphi'})\,d\varphi' = \frac{2}{\pi}\int_0^\infty \frac{\sinh(\pi x)}{x} K_{ix}(a) K_{ix}(b)\,dx$$

From Grad. + Ryzh., page 773, Eq. (10) the last integral exists, as follows:

$$\int_{-\pi/2}^{\pi/2} K_0(\sqrt{a^2+b^2-2ab\sin\varphi'})\,d\varphi' = \begin{cases} \pi I_0(a) K_0(b); & a<b \\ \pi I_0(b) K_0(a); & a>b \end{cases}$$

Consequently, in the case of $k_0 < \kappa$, we'll have:

$$\vec{A}(\vec{r},t) = \mu_0 J_{s0} R \hat{z} \begin{cases} K_0(\sqrt{\kappa^2-k_0^2}\,R)\, I_0(\sqrt{\kappa^2-k_0^2}\,\rho) \sin(\omega t - \kappa z); & \rho \le R; \text{ inside} \\ I_0(\sqrt{\kappa^2-k_0^2}\,R)\, K_0(\sqrt{\kappa^2-k_0^2}\,\rho) \sin(\omega t - \kappa z); & \rho \ge R; \text{ outside} \end{cases}$$

In the above equations $I_0(x) = J_0(ix)$ and $K_0(x) = i\frac{\pi}{2} H_0^{(1)}(ix) = i\frac{\pi}{2}[J_0(ix) + iY_0(ix)]$.
Both $I_0(x)$ and $K_0(x)$ are real-valued functions of the real-variable x.

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}(\vec{r},t) = -\frac{\partial A_z}{\mu_0 \partial \rho} \hat{\phi}$$

Note: $I_0'(x) = I_1(x)$, $K_0'(x) = -K_1(x)$

$I_0(x)K_1(x) + I_1(x)K_0(x) = 1/x$

Case I: $k_0 > \kappa$:

$$\vec{H}(\vec{r},t) = \frac{-1}{2}\pi J_{s0} R \hat{\phi} \sqrt{k_0^2-\kappa^2} \begin{cases} [Y_0(\sqrt{k_0^2-\kappa^2}\,R)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}\,R)\cos(\omega t-\kappa z)]\, J_1(\sqrt{k_0^2-\kappa^2}\,\rho); & \text{inside} \\ J_0(\sqrt{k_0^2-\kappa^2}\,R)[Y_1(\sqrt{k_0^2-\kappa^2}\,\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{k_0^2-\kappa^2}\,\rho)\cos(\omega t-\kappa z)]; & \text{outside} \end{cases}$$

Discontinuity of H_ϕ at the cylinder surface $= \frac{1}{2}\pi J_{s0}\sqrt{k_0^2-\kappa^2}\, R\, [Y_0(\cdots)J_1(\cdots) - Y_1(\cdots)J_0(\cdots)] \sin(\omega t-\kappa z)$

$$= \frac{\pi}{2} J_{s0} \sqrt{k_0^2-\kappa^2}\, R \frac{2}{\pi\sqrt{k_0^2-\kappa^2}\,R} \sin(\omega t-\kappa z) = J_{s0}\sin(\omega t-\kappa z) \;\hat{=}\; \text{Surface Current density} \checkmark$$

Case II: $k_0 < \kappa$

$$\vec{H}(\vec{r},t) = -J_{s0} R \sqrt{\kappa^2-k_0^2}\,\hat{\phi} \begin{cases} K_0(\sqrt{\kappa^2-k_0^2}\,R)\, I_1(\sqrt{\kappa^2-k_0^2}\,\rho)\sin(\omega t-\kappa z); & \text{inside} \\ -I_0(\sqrt{\kappa^2-k_0^2}\,R)\, K_1(\sqrt{\kappa^2-k_0^2}\,\rho)\sin(\omega t-\kappa z); & \text{outside} \end{cases}$$

Discontinuity of H_ϕ at cylinder surface $= J_{s0} R\sqrt{\kappa^2-k_0^2}\,[I_0(\sqrt{\cdots})K_1(\sqrt{\cdots}) + K_0(\sqrt{\cdots})I_1(\sqrt{\cdots})]\sin(\omega t-\kappa z)$
$= J_{s0}\sin(\omega t-\kappa z) =$ Surface Current density ✓

Computing the E-field from the H-field:

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} \Rightarrow -\frac{\partial H_\phi}{\partial z}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\hat{z} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow$$

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\left[-\kappa Y_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z) + \kappa J_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z)\right] J_1(\sqrt{k_0^2-\kappa^2}\rho)\hat{\rho}$$

$$-\frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\left[Y_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z)\right]\left[\frac{1}{\rho}J_1(\sqrt{k_0^2-\kappa^2}\rho)\right.$$

$$\left. + \sqrt{k_0^2-\kappa^2}\, J_1'(\sqrt{k_0^2-\kappa^2}\rho)\right]\hat{z} \quad \leftarrow k_0 > \kappa \text{ and } \rho < R \text{ (inside cylinder)}$$

$$\Rightarrow \epsilon_0 \vec{E}(\vec{r},t) = -\frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\frac{\kappa}{\omega}\left[Y_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z)\right]J_1(\sqrt{k_0^2-\kappa^2}\rho)\hat{\rho}$$

$$+\frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\frac{1}{\omega}\left[Y_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z)\right]\sqrt{k_0^2-\kappa^2}\,J_0(\sqrt{k_0^2-\kappa^2}\rho)\hat{z}$$

$$\Rightarrow \vec{E}(\vec{r},t) = -\frac{\pi}{2}Z_0 J_{s_0} R\kappa\sqrt{1-(\kappa/k_0)^2}\left[Y_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z) + J_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z)\right]J_1(\sqrt{k_0^2-\kappa^2}\rho)\hat{\rho}$$

$$+\frac{\pi}{2}Z_0 J_{s_0} R k_0\left(1-\frac{\kappa^2}{k_0^2}\right)\left[Y_0(\sqrt{k_0^2-\kappa^2}R)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}R)\sin(\omega t-\kappa z)\right]J_0(\sqrt{k_0^2-\kappa^2}\rho)\hat{z}$$

$k_0 > \kappa$ and $\rho < R$

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\,J_0(\sqrt{k_0^2-\kappa^2}R)\left[-\kappa Y_1(\sqrt{\cdots}\rho)\cos(\omega t-\kappa z) + \kappa J_1(\sqrt{\cdots}\rho)\sin(\omega t-\kappa z)\right]\hat{\rho}$$

$$-\frac{1}{2}\pi J_{s_0} R\sqrt{k_0^2-\kappa^2}\,J_0(\sqrt{\cdots}R)\sqrt{k_0^2-\kappa^2}\left[Y_0(\sqrt{\cdots}\rho)\sin(\omega t-\kappa z) + J_0(\sqrt{\cdots}\rho)\cos(\omega t-\kappa z)\right]\hat{z} \Rightarrow$$

$$\vec{E}(\vec{r},t) = -\frac{\pi}{2}Z_0 J_{s_0} R\kappa\sqrt{1-(\kappa/k_0)^2}\,J_0(\sqrt{k_0^2-\kappa^2}R)\left[Y_1(\sqrt{\cdots}\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{\cdots}\rho)\cos(\omega t-\kappa z)\right]\hat{\rho}$$

$$+\frac{\pi}{2}Z_0 J_{s_0} R k_0\left(1-\frac{\kappa^2}{k_0^2}\right)J_0(\sqrt{\cdots}R)\left[Y_0(\sqrt{\cdots}\rho)\cos(\omega t-\kappa z) - J_0(\sqrt{\cdots}\rho)\sin(\omega t-\kappa z)\right]\hat{z}$$

$k_0 > \kappa$ and $\rho \geq R$

Clearly E_z at the cylinder surface is continuous. The discontinuity of E_ρ is given by:

$$E_\rho(\text{outside}) - E_\rho(\text{inside}) = -\frac{\pi}{2}Z_0 J_{s_0} R\kappa\sqrt{1-(\kappa/k_0)^2}\left[J_0(\sqrt{\cdots}R)Y_1(\sqrt{\cdots}R) - J_1(\sqrt{\cdots}R)Y_0(\sqrt{\cdots}R)\right]\sin(\omega t-\kappa z)$$

$$= Z_0 J_{s_0}(\kappa/k_0)\sin(\omega t-\kappa z).$$

The $\vec{E}$ field discontinuity at the cylinder surface is directly related to the surface charge density, namely, $\vec{\nabla}\cdot\vec{J}_s + \frac{\partial \sigma_s}{\partial t} = 0 \Rightarrow \frac{\partial J_s}{\partial z} = -\frac{\partial \sigma_s}{\partial t} \Rightarrow$

$$\frac{\partial \sigma_s}{\partial t} = -J_{s0}\frac{\partial}{\partial z}\sin(\omega t - Kz) = K J_{s0}\cos(\omega t - Kz) \Rightarrow \sigma_s = \frac{K}{\omega} J_{s0}\sin(\omega t - Kz) \Rightarrow$$

$$\sigma_s/\epsilon_0 = \left(\frac{K}{k_0}\right)\frac{k_0}{\omega\epsilon_0} J_{s0}\sin(\omega t - Kz) = Z_0 J_{s0}(K/k_0)\sin(\omega t - Kz) \quad ✓$$

Case of $k_0 < K$:

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = J_{s0} R\sqrt{K^2-k_0^2}\left[-K K_0(\sqrt{\cdots}\,R) I_1(\sqrt{\cdots}\,\rho)\cos(\omega t - Kz)\right]\hat{\rho}$$

$$- J_{s0} R\sqrt{K^2-k_0^2}\, K_0(\sqrt{\cdots}\,R)\sin(\omega t - Kz)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho I_1(\sqrt{\cdots}\,\rho)\right]\hat{z}; \quad \rho < R \text{ (inside)}$$

$$I_1'(x) = I_0(x) - \frac{1}{x} I_1(x)$$

$$\Rightarrow \vec{E}(\vec{r},t) = -Z_0 J_{s0} R K\sqrt{(K/k_0)^2-1}\; K_0(\sqrt{\cdots}\,R) I_1(\sqrt{\cdots}\,\rho)\sin(\omega t - Kz)\hat{\rho}$$

$$+ Z_0 J_{s0} R k_0\left(\frac{K^2}{k_0^2}-1\right)\cos(\omega t - Kz)\, K_0(\sqrt{K^2-k_0^2}\,R)\, I_0(\sqrt{K^2-k_0^2}\,\rho)\hat{z}; \quad \begin{matrix} k_0<K \\ \rho<R \end{matrix}$$

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = J_{s0} R\sqrt{K^2-k_0^2}\left[K I_0(\sqrt{\cdots}\,R) K_1(\sqrt{\cdots}\,\rho)\cos(\omega t - Kz)\right]\hat{\rho}$$

$$+ J_{s0} R\sqrt{K^2-k_0^2}\; I_0(\sqrt{\cdots}\,R)\sin(\omega t - Kz)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho K_1(\sqrt{K^2-k_0^2}\,\rho)\right]\hat{z}; \quad \rho > R \text{ (outside)}$$

$$K_1'(x) = -K_0(x) - \frac{1}{x} K_1(x)$$

$$\Rightarrow \vec{E}(\vec{r},t) = Z_0 J_{s0} R K\sqrt{(K/k_0)^2-1}\; I_0(\sqrt{\cdots}\,R) K_1(\sqrt{\cdots}\,\rho)\sin(\omega t - Kz)\hat{\rho}$$

$$+ Z_0 J_{s0} R k_0\left(\frac{K^2}{k_0^2}-1\right) I_0(\sqrt{\cdots}\,R)\cos(\omega t - Kz)\, K_0(\sqrt{\cdots}\,\rho)\hat{z}; \quad \begin{matrix} k_0<K \\ \rho>R \text{ (outside)} \end{matrix}$$

Again, E_z is continuous at the surface. The discontinuity of E_ρ is given by

$$E_\rho(\text{outside}) - E_\rho(\text{inside}) = Z_0 J_{s0} R K\sqrt{(K/k_0)^2-1}\left[I_0(\sqrt{\cdots})K_1(\sqrt{\cdots}) + K_0(\sqrt{\cdots}) I_1(\sqrt{\cdots})\right]\sin(\omega t - Kz)$$

$$= Z_0 J_{s0}(K/k_0)\sin(\omega t - Kz) \quad ✓ \quad (\text{Same as } \sigma_s/\epsilon_0)$$

Scalar potential $\psi(\vec{r},t)$ is derived from $J_s(\vec{r}',t) = \frac{K}{\omega} J_{s0} \sin(\omega t - K z')$. The equations are similar to those derived for $\vec{A}(\vec{r},t)$. Therefore,

$$\psi(\vec{r},t) = \frac{-\pi (K/\omega) J_{s0} R}{2\epsilon_0} \begin{cases} [Y_0(\sqrt{k_0^2-K^2}R)\sin(\omega t - Kz) + J_0(\sqrt{k_0^2-K^2}R)\cos(\omega t - Kz)] J_0(\sqrt{k_0^2-K^2}\rho), & \text{inside} \\ J_0(\sqrt{k_0^2-K^2}R)[Y_0(\sqrt{k_0^2-K^2}\rho)\sin(\omega t - Kz) + J_0(\sqrt{k_0^2-K^2}\rho)\cos(\omega t - Kz)], & \text{outside} \end{cases}$$

$$\vec{\psi}(\vec{r},t) = \frac{(K/\omega) J_{s0} R}{\epsilon_0} \begin{cases} K_0(\sqrt{K^2-k_0^2}R)\, I_0(\sqrt{K^2-k_0^2}\rho)\sin(\omega t - Kz); & \rho \le R,\ k_0 < K \\ I_0(\sqrt{K^2-k_0^2}R)\, K_0(\sqrt{K^2-k_0^2}\rho)\sin(\omega t - Kz) & \rho \ge R,\ k_0 < K \end{cases}$$

The $\vec{E}$-field can now be derived directly using $\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t}$.

$$\vec{E}(\vec{r},t) = -\frac{\partial \psi}{\partial \rho}\hat{\rho} - \left(\frac{\partial \psi}{\partial z} + \frac{\partial A_z}{\partial t}\right)\hat{z}.$$

$$\vec{E}(\vec{r},t) = -\frac{\pi}{2} Z_0 J_{s0} R K \sqrt{1-(K/k_0)^2}\, [Y_0(\sqrt{\cdots}R)\sin(\omega t - Kz) + J_0(\sqrt{\cdots}R)\cos(\omega t - Kz)] J_1(\sqrt{\cdots}\rho)\,\hat{\rho}$$

$$-\left\{\frac{\pi(K^2/\omega)J_{s0}R}{2\epsilon_0}[Y_0(\sqrt{\cdots}R)\cos(\omega t - Kz) - J_0(\sqrt{\cdots}R)\sin(\omega t - Kz)] J_0(\sqrt{\cdots}\rho)\right.$$

$$\left. -\frac{1}{2}\mu_0 \pi J_{s0} R\omega [Y_0(\sqrt{\cdots}R)\cos(\omega t - Kz) - J_0(\sqrt{\cdots}R)\sin(\omega t - Kz)] J_0(\sqrt{\cdots}\rho)\right\}\hat{z}; \quad \begin{matrix} k_0 > K \\ \rho < R \end{matrix}$$

The last two terms can be combined to yield the same relationship for $\vec{E}(\vec{r},t)$ as before. In the same way, all the other cases can also be confirmed. ✓

Poynting Vector: (i) The case of $k_0 > K$, inside the cylinder ($\rho < R$):

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \left(-\frac{\pi}{2} Z_0 J_{s0} R K \sqrt{1-(K/k_0)^2}\right)\left(-\frac{\pi}{2} J_{s0} R \sqrt{k_0^2-K^2}\right)\hat{z}\,[Y_0(\sqrt{\cdots}R)\sin(\cdots)$$

$$+ J_0(\sqrt{\cdots}R)\cos(\cdots)][Y_0(\sqrt{\cdots}R)\sin(\cdots) + J_0(\sqrt{\cdots}R)\cos(\cdots)]\, J_1^2(\sqrt{\cdots}\rho)$$

$$+\left[\frac{\pi}{2} Z_0 J_{s0} R k_0 \left(1-\frac{K^2}{k_0^2}\right)\right]\left[+\frac{\pi}{2} J_{s0} R\sqrt{k_0^2-K^2}\right]\hat{\rho}\,[Y_0(\sqrt{\cdots}R)\cos(\cdots) - J_0(\sqrt{\cdots}R)\sin(\cdots)][Y_0(\sqrt{\cdots}R)\sin(\cdots)$$

$$+J_0(\sqrt{}R)\,Co(-)]\,J_0(\sqrt{\cdots}\rho)\,J_1(\sqrt{\cdots}\rho) \Rightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\pi^2}{4} Z_0 J_{s0}^2 R^2 k_0 K \left(1-\frac{K^2}{k_0^2}\right)\left[\frac{1}{2}Y_0^2(\sqrt{\cdots}R)+\frac{1}{2}J_0^2(\sqrt{\cdots}R)\right]J_1^2(\sqrt{\cdots}\rho)\,\hat{z} \Rightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{32} Z_0 (2\pi R J_{s0})^2 (K/k_0)(k_0^2-K^2)\left[Y_0^2(\sqrt{k_0^2-K^2}R)+J_0^2(\sqrt{k_0^2-K^2}R)\right]J_1^2(\sqrt{k_0^2-K^2}\rho)\,\hat{z}$$

$k_0 > K$: inside cylinder

(ii) Case of $k_0 > K$, outside the cylinder ($\rho > R$):

$$\vec{S}(\vec{r},t) = \left(-\frac{\pi}{2} Z_0 J_{s0} R K\sqrt{1-K^2/k_0^2}\right)\left(-\frac{\pi}{2}J_{s0}R\sqrt{k_0^2-K^2}\right)J_0^2(\sqrt{\cdots}R)\left[Y_1(\sqrt{}\rho)\,\sin(\cdots)+\right.$$
$$\left.J_1(\sqrt{\cdots}\rho)\,Co(-)\right]\left[Y_1(\sqrt{}\rho)\sin(-)+J_1(\sqrt{\cdots}\rho)\,Co(-)\right]\hat{z}$$

$$+\left[\frac{\pi}{2} Z_0 J_{s0} R k_0 \left(1-\frac{K^2}{k_0^2}\right)\right]\left(+\frac{1}{2}\pi J_{s0} R\sqrt{k_0^2-K^2}\right)J_0^2(\sqrt{\cdots}R)\left[Y_0(\sqrt{\cdots}\rho)\,Co(-)-J_0(\sqrt{\cdots}\rho)\sin(-)\right]$$
$$\left[Y_1(\sqrt{}\rho)\sin(-)+J_1(\sqrt{}\rho)\,Co(-)\right]\hat{\rho} \Rightarrow$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{\pi^2}{4} Z_0 J_{s0}^2 R^2 (K/k_0)(k_0^2-K^2)\,J_0^2(\sqrt{k_0^2-K^2}R)\left[\frac{1}{2}Y_1^2(\sqrt{\cdots}\rho)+\frac{1}{2}J_1^2(\sqrt{\cdots}\rho)\right]\hat{z}$$

$$+\frac{\pi^2}{4} Z_0 J_{s0}^2 R^2 k_0^2 \left(1-\frac{K^2}{k_0^2}\right)^{3/2} J_0^2(\sqrt{\cdots}R)\left[\frac{1}{2}Y_0(\sqrt{\cdots}\rho)J_1(\sqrt{\cdots}\rho)-\frac{1}{2}J_0(\sqrt{\cdots}\rho)Y_1(\sqrt{\cdots}\rho)\right]\hat{\rho}$$

$$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{32} Z_0 (2\pi R J_{s0})^2 (K/k_0)(k_0^2-K^2)\,J_0^2(\sqrt{k_0^2-K^2}R)\left[Y_1^2(\sqrt{k_0^2-K^2}\rho)+J_1^2(\sqrt{k_0^2-K^2}\rho)\right]\hat{z}$$

$$+\frac{1}{16\pi} Z_0 (2\pi R J_{s0})^2 \left(\frac{k_0}{\rho}\right)\left(1-\frac{K^2}{k_0^2}\right)J_0^2(\sqrt{k_0^2-K^2}R)\,\hat{\rho}\,;$$

$k_0 > K$: outside cylinder

(iii) Case of $k_0 < K$, inside the cylinder ($\rho < R$):

$$\vec{S}(\vec{r},t) = \left(-Z_0 J_{s0} R K\sqrt{(K/k_0)^2-1}\right)\left(-J_{s0}R\sqrt{K^2-k_0^2}\right)K_0^2(\sqrt{\cdots}R)\,I_1^2(\sqrt{\cdots}\rho)\,\sin^2(-)\,\hat{z}$$

$$+\left[Z_0 J_{s0} R k_0\left(\frac{K^2}{k_0^2}-1\right)\right]\left(+J_{s0}R\sqrt{K^2-k_0^2}\right)K_0^2(\sqrt{\cdots}R)\,I_0(\sqrt{\cdots}\rho)\,I_1(\sqrt{\cdots}\rho)\sin(\cdots)\,Co(\cdots)\,\hat{\rho}$$

$$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2} Z_0 J_{s0}^2 R^2 (K/k_0)(K^2-k_0^2)\,K_0^2(\sqrt{K^2-k_0^2}R)\,I_1^2(\sqrt{K^2-k_0^2}\rho)\,\hat{z}\;;\text{ inside}$$

(iv) Case of $k_0 < K$, outside the cylinder ($\rho > R$):

$$\vec{S}(\vec{r},t) = \left(Z_0 J_{s0} R K\sqrt{K^2/k_0^2-1}\right)\left(J_{s0}R\sqrt{K^2-k_0^2}\right)I_0^2(\sqrt{\cdots}R)\,K_1^2(\sqrt{\cdots}\rho)\,\sin^2(\cdots)\,\hat{z}$$

$$+\left[Z_0 J_{s_0} R k_0 \left(\frac{K^2}{k_0^2}-1\right)\right]\left[-J_{s_0} R\sqrt{K^2-k_0^2}\right] I_0^2(\sqrt{\cdots}\,R) K_0(\sqrt{\cdots}\,\rho) K_1(\sqrt{\cdots}\,\rho) \sin(\ldots)\cos(\ldots)\hat{\rho}$$

$$\Rightarrow \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2} Z_0 J_{s_0}^2 R^2 (K/k_0)(K^2-k_0^2) I_0^2(\sqrt{K^2-k_0^2}\,R) K_1^2(\sqrt{K^2-k_0^2}\,\rho)\hat{z} \text{ ; outside}$$

Therefore, when $k_0 < K$, the field outside the cylinder is an evanescent field that does not carry any energy in the radial direction. All the energy moves up along the z-axis. This is like a transmission line which does not radiate any of the electromagnetic energy away. Rather, it carries the energy along the length of the wire.

Conservation of energy: In the case of $k_0 > K$, the total radiated power in the radial direction is $2\pi\rho\langle S_\rho(\rho,t)\rangle$ (per unit length of the cylinder). This may be written as,

$$\text{Radiated power} = \frac{1}{8} Z_0 (2\pi R J_{s_0})^2 k_0 \left(1-\frac{K^2}{k_0^2}\right) J_0^2(\sqrt{k_0^2-K^2}\,R)$$

On the other hand, the $\vec{E}$-field acting on the cylinder surface, is:

$$E_z(\rho=R,t) = \frac{\pi}{2} Z_0 J_{s_0} R k_0 \left(1-\frac{K^2}{k_0^2}\right) J_0(\sqrt{\cdots}\,R)\left[Y_0(\sqrt{\cdots}\,R)\cos(\omega t\ldots) - J_0(\sqrt{\cdots}\,R)\sin(\ldots)\right]$$

This must be multiplied by $2\pi R\, J_{s_0} \sin(\omega t - Kz)$, then time averaged to yield the radiated power. The result is:

$$\pi^2 R^2 Z_0 J_{s_0}^2 k_0 \left(1-\frac{K^2}{k_0^2}\right) J_0^2(\sqrt{\cdots}\,R)\langle \sin^2(\ldots)\rangle = \frac{1}{8} Z_0 (2\pi R J_{s_0})^2 k_0 \left(1-\frac{K^2}{k_0^2}\right) J_0^2(\sqrt{\cdots}\,R)$$

in agreement with the earlier result (Total radiated power).