Solutions **Opti 501**

Problem 32)

a) Since $\vec{\nabla}\cdot\vec{J}=0$ we have $\rho(r',t)=0$; therefore, $\psi(\vec{r},t)=0$. Consequently,

$$\vec{E}(\vec{r},t)=-\frac{\partial\vec{A}}{\partial t}=\frac{1}{4}\mu_0 I_0\omega\begin{cases}J_0(k_0R)[Y_0(k_0\rho)\cos\omega t-J_0(k_0\rho)\sin\omega t]\hat{z}; & \rho\geq R\\ [Y_0(k_0R)\cos\omega t-J_0(k_0R)\sin\omega t]J_0(k_0\rho)\hat{z}; & \rho\leq R\end{cases}$$

Since $\omega=k_0c$ and $\mu_0c=Z_0$, the coefficient of the above expression may be written as $\frac{1}{4}\mu_0 I_0\omega=\frac{1}{4}Z_0I_0k_0$, which has units of Volts/meter.

$$\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}=-\frac{\partial A_z}{\partial\rho}\hat{\phi}=-\frac{\mu_0I_0k_0}{4}\begin{cases}J_0(k_0R)[Y_1(k_0\rho)\sin\omega t+J_1(k_0\rho)\cos\omega t]\hat{\phi}; & \rho\geq R\\ [Y_0(k_0R)\sin\omega t+J_0(k_0R)\cos\omega t]J_1(k_0\rho)\hat{\phi}; & \rho\leq R\end{cases}$$

Here we have used the relations $J_0'(\cdot)=-J_1(\cdot)$ and $Y_0'(\cdot)=-Y_1(\cdot)$.

b) $\vec{H}(r=R^+,t)-\vec{H}(r=R^-,t)=\frac{I_0k_0}{4}\left[Y_0(k_0R)J_1(k_0R)-Y_1(k_0R)J_0(k_0R)\right]\sin\omega t\,\hat{\phi}.$

The notes on Bessel functions (posted to the website) contain a relationship between Bessel functions as follows: $Y_n(x)J_{n+1}(x)-J_n(x)Y_{n+1}(x)=\frac{2}{\pi x}$. Thus

$$\vec{H}(R^+,t)-\vec{H}(R^-,t)=\frac{I_0k_0}{4}\,\frac{2}{\pi k_0R}\sin(\omega t)\hat{\phi}=\frac{I_0}{2\pi R}\sin(\omega t)\hat{\phi}=J_{s_0}\sin(\omega t)\hat{\phi}\ \checkmark$$

C) Let $R\gg\lambda_0$ and $\rho=R+x$ for points outside the cylinder, while $\rho=R-x$ for points inside the cylinder. Thus $x>0$ is the distance of the observation point from the cylinder wall. We also assume that $x\ll R$, so that the observation point is not too far from the cylinder's wall. We thus have $k_0R\gg1$ and $k_0\rho=k_0(R\pm x)\gg1$. The limiting form of the Bessel functions for large argument is as follows:

$$J_n(x)\rightarrow\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right);\qquad Y_n(x)\rightarrow\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right).$$

Therefore, the $\vec{E}$- and $\vec{H}$-fields in the vicinity of a cylinder of large radius will be:

$$\vec{E}(x,t)\cong\frac{Z_0I_0k_0}{4}\begin{cases}\frac{2}{\pi k_0\sqrt{R(R+x)}}\cos(k_0R-\frac{\pi}{4})\left[\sin(k_0R+k_0x-\frac{\pi}{4})\cos\omega t-\cos(k_0R+k_0x-\frac{\pi}{4})\sin\omega t\right]\hat{z}; & \text{outside}\\ \frac{2}{\pi k_0\sqrt{R(R-x)}}\left[\sin(k_0R-\frac{\pi}{4})\cos\omega t-\cos(k_0R-\frac{\pi}{4})\sin\omega t\right]\cos(k_0R-k_0x-\frac{\pi}{4})\hat{z}; & \text{inside}\end{cases}$$

$$\vec{E}(x,t) \cong \frac{Z_0 I_0}{2\pi R} \begin{cases} \left(1+\frac{x}{R}\right)^{-1/2} \cos\left(k_0R-\frac{\pi}{4}\right) \sin\left(k_0R+k_0x-\omega t-\pi/4\right)\hat{z}; & \text{outside} \\ \left(1-\frac{x}{R}\right)^{-1/2} \sin\left(k_0R-\omega t-\pi/4\right)\cos\left(k_0R-k_0x-\frac{\pi}{4}\right)\hat{z}; & \text{inside} \end{cases} \Rightarrow$$

$$\vec{E}(x,t) \simeq Z_0 J_{s0} \begin{cases} \left(1-\frac{x}{2R}\right)\left[\frac{1}{2}\sin\left(k_0(2R+x)-\omega t-\frac{\pi}{2}\right)+\frac{1}{2}\sin(k_0x-\omega t)\right]\hat{z}; & \text{outside} \\ \left(1+\frac{x}{2R}\right)\left[\frac{1}{2}\sin\left(k_0(2R-x)-\omega t-\frac{\pi}{2}\right)+\frac{1}{2}\sin(k_0x-\omega t)\right]\hat{z}; & \text{inside} \end{cases} \Rightarrow$$

$$\vec{E}(x,t) \cong \frac{1}{2} Z_0 J_{s0}\left(1 \mp \frac{x}{2R}\right)\left\{\sin(k_0x-\omega t) - \cos\left[k_0(2R\pm x)-\omega t\right]\right\}\hat{z}; \ (+\text{outside}, -\text{inside.})$$

In the limit $R\to\infty$ the term $\mp\frac{x}{2R}$ can be set to zero. The second term containing $2R\pm x$ contains the contributions of the far-away walls of the cylinder. Therefore, the $\vec{E}$-field produced by the wall of the cylinder that is near the observation point is given by $\vec{E}_{near}(x,t) \cong \frac{1}{2} Z_0 J_{s0} \sin(k_0x-\omega t)\hat{z}$.

Similarly the magnetic field $\vec{H}(\vec{r},t)$ in the vicinity of the wall of a large cylinder ($R \gg \lambda_0$) may be written as:

$$\vec{H}(x,t) \cong -\frac{I_0 k_0}{4} \begin{cases} \frac{2}{\pi k_0\sqrt{R(R+x)}}\cos\left(k_0R-\frac{\pi}{4}\right)\left[\sin\left(k_0R+k_0x-\frac{3\pi}{4}\right)\sin\omega t+\cos\left(k_0R+k_0x-\frac{3\pi}{4}\right)\cos\omega t\right]\hat{\phi} \\ \frac{2}{\pi k_0\sqrt{R(R-x)}}\left[\sin\left(k_0R-\frac{\pi}{4}\right)\sin\omega t+\cos\left(k_0R-\frac{\pi}{4}\right)\cos\omega t\right]\cos\left(k_0R-k_0x-\frac{3\pi}{4}\right)\hat{\phi} \end{cases}$$

$$\simeq -\frac{I_0}{2\pi R} \begin{cases} \left(1+\frac{x}{R}\right)^{-1/2}\cos\left(k_0R-\frac{\pi}{4}\right)\cos\left(k_0R+k_0x-\omega t-\frac{3\pi}{4}\right)\hat{\phi}; & \text{outside} \\ \left(1-\frac{x}{R}\right)^{-1/2}\cos\left(k_0R-\omega t-\frac{\pi}{4}\right)\cos\left(k_0R-k_0x-\frac{3\pi}{4}\right)\hat{\phi}; & \text{inside} \end{cases}$$

$$\cong -J_{s0} \begin{cases} \left(1-\frac{x}{2R}\right)\left\{\frac{1}{2}\cos\left[k_0(2R+x)-\omega t-\pi\right]+\frac{1}{2}\cos\left(k_0x-\omega t-\frac{\pi}{2}\right)\right\}\hat{\phi}; & \text{outside} \\ \left(1+\frac{x}{2R}\right)\left\{\frac{1}{2}\cos\left[k_0(2R-x)-\omega t-\pi\right]+\frac{1}{2}\cos\left(k_0x-\omega t+\frac{\pi}{2}\right)\right\}\hat{\phi}; & \text{inside} \end{cases} \Rightarrow$$

$$\vec{H}(x,t) = \frac{1}{2} J_{s0}\left(1\mp\frac{x}{2R}\right)\left\{\mp\sin(k_0x-\omega t)+\cos\left[k_0(2R\pm x)-\omega t\right]\right\}\hat{\phi}; \ (+\text{outside}, -\text{inside.})$$

Again, in the limit $R\to\infty$, the term $\mp\frac{x}{2R}$ can be set to zero. The second term containing $2R\pm x$ contains the contributions of the far-away walls of the cylinder. The wall of the cylinder near the observation point thus produces the $\vec{H}$-field $\vec{H}_{near}(x,t) \simeq \mp\frac{1}{2} J_{s0}\sin(k_0x-\omega t)\hat{\phi}$.