Problem 30)

a) In cylindrical coordinates, the current-density is in the $\hat{\boldsymbol{\phi}}$ direction and depends on ρ and z, but *not* on ϕ, that is, $\boldsymbol{J}(\boldsymbol{r},t) = J_\phi(\rho,z,t)\hat{\boldsymbol{\phi}}$. Therefore, $\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r},t) = \rho^{-1}(\partial J_\phi/\partial\phi) = 0$. The charge-current continuity equation now yields $\partial\rho/\partial t = 0$, and since no charges are placed *a priori* on the loop, we conclude that $\rho(\boldsymbol{r},t) = 0$ everywhere in space.

b) The two segments of the loop that are parallel to $\hat{\boldsymbol{y}}$ are equidistant from the observation point $\boldsymbol{r} = y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, but their currents are in opposite directions. Therefore, their contributions to $\boldsymbol{A}(\boldsymbol{r},t)$ cancel out. The remaining two segments have equal lengths ($\tfrac{1}{2}\pi R$ each), are centered at $(x',y',z') = (0,\pm R,0)$, and carry current along $\mp\hat{\boldsymbol{x}}$. (Note that we are essentially approximating the circular loop with a square loop.) For the latter two segments, therefore,

$$\boldsymbol{J}(\boldsymbol{r}',t)d\boldsymbol{r}' = \mp(\tfrac{1}{2}\pi R)I_0\cos(2\pi f t)\,\hat{\boldsymbol{x}} \qquad (\text{at } \boldsymbol{r}' = \pm R\hat{\boldsymbol{y}}).$$

Using approximate expressions for the square roots, namely, $\sqrt{1+\varepsilon} \cong 1 + \tfrac{1}{2}\varepsilon$ and $1/\sqrt{1+\varepsilon} = (1+\varepsilon)^{-1/2} \cong 1 - \tfrac{1}{2}\varepsilon$, where $|\varepsilon| \ll 1.0$, we may write

$$|\boldsymbol{r}-\boldsymbol{r}'| = |y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \mp R\hat{\boldsymbol{y}}| = \sqrt{(y\mp R)^2 + z^2} = \sqrt{(y^2+z^2) + R^2 \mp 2Ry}$$

R^2, being smaller than the other terms, is dropped.

$$\cong \sqrt{r^2 \mp 2Ry} = r\sqrt{1 \mp 2(y/r^2)R} \cong r[1 \mp (y/r^2)R] = r \mp (y/r)R = r \mp R\sin\theta.$$

Similarly,

$$|\boldsymbol{r}-\boldsymbol{r}'|^{-1} = |y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \mp R\hat{\boldsymbol{y}}|^{-1} = [(y\mp R)^2 + z^2]^{-1/2} = [(y^2+z^2) + R^2 \mp 2Ry]^{-1/2}$$

$$\cong [r^2 \mp 2Ry]^{-1/2} = r^{-1}[1 \mp 2(y/r^2)R]^{-1/2} \cong r^{-1}[1 \pm (y/r^2)R] = \frac{1}{r}\left(1 \pm \frac{R\sin\theta}{r}\right).$$

The vector potential is thus given by

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\iiint_{-\infty}^{\infty}\frac{\boldsymbol{J}(\boldsymbol{r}',\ t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'$$

$$= \frac{\mu_0}{4\pi}(\tfrac{1}{2}\pi R I_0)\left\{-\frac{\cos[2\pi f(t-|\boldsymbol{r}-\boldsymbol{r}_1'|/c)]}{|\boldsymbol{r}-\boldsymbol{r}_1'|} + \frac{\cos[2\pi f(t-|\boldsymbol{r}-\boldsymbol{r}_2'|/c)]}{|\boldsymbol{r}-\boldsymbol{r}_2'|}\right\}\hat{\boldsymbol{x}}$$

$$= -\frac{\mu_0\pi R I_0}{8\pi r}\left\{\left(1+\frac{R\sin\theta}{r}\right)\cos\left[2\pi f\left(t-\frac{r-R\sin\theta}{c}\right)\right] - \left(1-\frac{R\sin\theta}{r}\right)\cos\left[2\pi f\left(t-\frac{r+R\sin\theta}{c}\right)\right]\right\}\hat{\boldsymbol{x}}.$$

The symmetry of the problem allows one to replace $-\hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{\phi}}$ of the spherical coordinate system. Combining similar terms and using the two identities

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right),$$

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

we now find

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0\pi R I_0}{4\pi r}\left\{-\sin\left[2\pi f\left(t-\frac{r}{c}\right)\right]\sin\left(\frac{2\pi f R\sin\theta}{c}\right)\right.$$

$$\left. + \left(\frac{R\sin\theta}{r}\right)\cos\left[2\pi f\left(t-\frac{r}{c}\right)\right]\cos\left(\frac{2\pi f R\sin\theta}{c}\right)\right\}\hat{\boldsymbol{\phi}}.$$

Considering that $(2\pi fR/c)\sin\theta = (2\pi R/\lambda_0)\sin\theta \ll 1.0$, where $R \ll \lambda_0 = c/f$ and λ_0 is the vacuum wavelength, we use the small angle approximation $\sin\alpha \approx \alpha$ and $\cos\alpha \approx 1 - ½\alpha^2$, then ignore terms of order R^2 and higher, to obtain

$$\boldsymbol{A}(\boldsymbol{r},t) \cong \left(\frac{\mu_0 \pi R^2 I_0}{4\pi}\right)\left\{-\left(\frac{2\pi}{\lambda_0}\right)\frac{\sin[2\pi f(t-r/c)]}{r} + \frac{\cos[2\pi f(t-r/c)]}{r^2}\right\}\sin\theta\,\hat{\boldsymbol{\phi}}.$$

Note that the coefficient $\mu_0\pi R^2 I_0$ is the magnitude m_0 of the magnetic dipole moment $\boldsymbol{m}(t)$.

c) $$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta}\hat{\boldsymbol{r}} - \frac{\partial(rA_\phi)}{r\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{m_0}{4\pi}(2\cos\theta)\left[-\left(\frac{2\pi}{\lambda_0}\right)\frac{1}{r^2}\sin(\cdots) + \frac{1}{r^3}\cos(\cdots)\right]\hat{\boldsymbol{r}}$$ ← "…" stands for $2\pi f(t - r/c)$.

$$-\frac{m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left[\left(\frac{2\pi}{\lambda_0}\right)^2\cos(\cdots) - \frac{1}{r^2}\cos(\cdots) + \left(\frac{2\pi}{\lambda_0}\right)\frac{1}{r}\sin(\cdots)\right]\hat{\boldsymbol{\theta}}$$

$$= \frac{m_0}{4\pi}\left\{\left[-\left(\frac{2\pi}{\lambda_0}\right)\frac{\sin(\cdots)}{r^2} + \frac{\cos(\cdots)}{r^3}\right](2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) - \left(\frac{2\pi}{\lambda_0}\right)^2\frac{\cos(\cdots)}{r}\sin\theta\,\hat{\boldsymbol{\theta}}\right\}.$$

Using $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$, the above B-field may be rewritten as

$$\boldsymbol{B}(\boldsymbol{r},t) = -\frac{m_0}{4\pi}\left(\frac{2\pi}{\lambda_0}\right)\left\{\left[\frac{\sin(\cdots)}{r^2} - \left(\frac{\lambda_0}{2\pi}\right)\frac{\cos(\cdots)}{r^3}\right](3\cos\theta\,\hat{\boldsymbol{r}} - \hat{\boldsymbol{z}}) + \left(\frac{2\pi}{\lambda_0}\right)\frac{\cos(\cdots)}{r}\sin\theta\,\hat{\boldsymbol{\theta}}\right\}.$$

Next we calculate the E-field, as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{A}}{\partial t} = \frac{m_0}{4\pi}\left\{\left(\frac{2\pi}{\lambda_0}\right)(2\pi f)\frac{\cos[2\pi f(t-r/c)]}{r} + (2\pi f)\frac{\sin[2\pi f(t-r/c)]}{r^2}\right\}\sin\theta\,\hat{\boldsymbol{\phi}}.$$

Using $f = c/\lambda_0$, the above equation may be written

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{m_0 c}{4\pi}\left(\frac{2\pi}{\lambda_0}\right)\left\{\left(\frac{2\pi}{\lambda_0}\right)\frac{\cos[2\pi f(t-r/c)]}{r} + \frac{\sin[2\pi f(t-r/c)]}{r^2}\right\}\sin\theta\,\hat{\boldsymbol{\phi}}.$$

As a check on the above calculations, we confirm that $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)$ is equal to $-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$, as required by Maxwell's third equation.

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}\hat{\boldsymbol{r}} - \frac{\partial(rE_\phi)}{r\partial r}\hat{\boldsymbol{\theta}}$$

$$= \frac{m_0 c}{4\pi}\left(\frac{2\pi}{\lambda_0}\right)(2\cos\theta)\left[\left(\frac{2\pi}{\lambda_0}\right)\frac{\cos(\cdots)}{r^2} + \frac{\sin(\cdots)}{r^3}\right]\hat{\boldsymbol{r}}$$ ← "…" stands for $2\pi f(t - r/c)$.

$$-\frac{m_0 c}{4\pi}\left(\frac{2\pi}{\lambda_0}\right)\left(\frac{\sin\theta}{r}\right)\left[\left(\frac{2\pi}{\lambda_0}\right)^2\sin(\cdots) - \frac{\sin(\cdots)}{r^2} - \left(\frac{2\pi}{\lambda_0}\right)\frac{\cos(\cdots)}{r}\right]\hat{\boldsymbol{\theta}}$$

$$= \frac{m_0 c}{4\pi}\left(\frac{2\pi}{\lambda_0}\right)\left\{\left[\left(\frac{2\pi}{\lambda_0}\right)\frac{\cos(\cdots)}{r^2} + \frac{\sin(\cdots)}{r^3}\right](3\cos\theta\,\hat{\boldsymbol{r}} - \hat{\boldsymbol{z}}) - \left(\frac{2\pi}{\lambda_0}\right)^2\frac{\sin(\cdots)}{r}\sin\theta\hat{\boldsymbol{\theta}}\right\}.$$

The equality of the above expression with $-\partial\boldsymbol{B}/\partial t$ is readily confirmed.
