**Solutions** **Opti 501**

Problem 29)

$$|\vec r-\vec r'| = |x\hat x + y\hat y + z\hat z \mp \tfrac14 d\hat z| = \left[x^2+y^2+(z\mp\tfrac{d}{4})^2\right]^{1/2}$$

$$= \sqrt{x^2+y^2+z^2}\left[1+\frac{\frac{1}{16}d^2 \mp \frac12 zd}{x^2+y^2+z^2}\right]^{1/2} \simeq r\left[1+\frac{(d^2/32)\mp(zd/4)}{r^2}\right]$$

$$= r \mp \frac{zd}{4r} + \frac{d^2}{32r} \Rightarrow |\vec r-\vec r'|\simeq r\mp\frac{zd}{4r}$$ ← To first order in $d$

Similarly, $|\vec r-\vec r'|^{-1} = \left[x^2+y^2+(z\mp\frac{d}{4})^2\right]^{-1/2} =$

$$(x^2+y^2+z^2)^{-1/2}\left[1+\frac{\frac{d^2}{16}\mp\frac12 dz}{x^2+y^2+z^2}\right]^{-1/2} \simeq \frac1r\left[1-\frac{(d^2/32)\mp(zd/4)}{r^2}\right]$$

$$= \frac1r \pm \frac{zd}{4r^3} - \frac{d^2}{32r^3} \Rightarrow \frac{1}{|\vec r-\vec r'|}\simeq\frac1r\left(1\pm\frac{zd}{4r^2}\right)$$ ← To first order in $d$

$$\vec A(\vec r,t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec J(\vec r', t-|r-r'|/c)}{|r-r'|}\,dv' \cong \frac{\mu_0 I_0 (d/2)}{4\pi}\left\{\frac{\sin[2\pi f(t-|r-r_1|/c)]}{|\vec r-\vec r_1|}\right.$$

$$\left. + \frac{\sin[2\pi f(t-|r-r_2|/c)]}{|\vec r-\vec r_2|}\right\}\hat z \qquad \text{where } r_{1,2}=\pm(\tfrac14 d)\hat z$$

$$\Rightarrow \vec A(\vec r,t)\simeq\frac{\mu_0 I_0 d}{8\pi r}\left\{\left(1+\frac{zd}{4r^2}\right)\sin\left[2\pi f\left(t-\frac rc+\frac{zd}{4rc}\right)\right]+\left(1-\frac{zd}{4r^2}\right)\sin\left[2\pi f\left(t-\frac rc-\frac{zd}{4rc}\right)\right]\right\}\hat z$$

Next we combine similar terms and use the identities $\sin a+\sin b = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$ and $\sin a - \sin b = 2\sin(\frac{a-b}{2})\cos(\frac{a+b}{2})$. We'll have

$$\vec A(\vec r,t) \simeq \frac{\mu_0 I_0 d}{4\pi r}\left\{\sin[2\pi f(t-r/c)]\cos\left(\frac{\pi f z d}{2rc}\right) + \left(\frac{zd}{4r^2}\right)\sin\left(\frac{\pi f zd}{2rc}\right)\cos[2\pi f(t-r/c)]\right\}\hat z$$

The above formula can be further simplified if one recognizes that $\frac{\pi f z d}{2rc} = (\frac{\pi d}{\lambda_0})\cos\theta \ll 1$ if $d\ll\lambda_0$. (Here $\lambda_0=c/f$ is the wavelength of the radiation.). For small $\alpha$, we expand $\sin\alpha$ and $\cos\alpha$ in Taylor series to find $\sin\alpha\simeq\alpha$ and $\cos\alpha\simeq 1-\frac12\alpha^2$. Therefore,

$$\vec A(\vec r,t) \simeq \frac{\mu_0 I_0 d}{4\pi r}\left\{\sin[2\pi f(t-r/c)] - \frac12\left(\frac{\pi d}{\lambda_0}\right)^2\cos^2\theta\left[\sin[2\pi f(t-r/c)] - \frac{\lambda_0}{2\pi r}\cos[2\pi f(t-r/c)]\right]\right\}\hat z$$

Note that the first term in the above expression is the same as that given in Problem 2, while the second term is a correction of order $(d/\lambda_0)^2$ that accounts for the finite size of $d$.