Solutions **Opti 501**

Problem 28)

Let $\omega = 2\pi f$, $\lambda_0 = c/f$, $k_0 = 2\pi/\lambda_0$, $c = 1/\sqrt{\mu_0\epsilon_0}$, $Z_0 = \sqrt{\mu_0/\epsilon_0}$.
We have shown (in the class) that, for $r \gg d$, the $\vec{E}$-field distribution is given by:

$$\vec{E}(\vec{r},t) = \frac{Z_0 I_0 d}{4\pi}\left\{\left[\frac{1}{r^2}\sin(\omega t - k_0 r) - \frac{\lambda_0}{2\pi r^3}\cos(\omega t - k_0 r)\right]\right.$$

multiplication sign

$$\left. \times (3\cos\theta\,\hat{r} - \hat{z}) + \frac{k_0}{r}\sin\theta\cos(\omega t - k_0 r)\,\hat{\theta}\right\}.$$

Here $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ is the unit vector along the $\hat{z}$-axis.

Taking the factor $-\lambda_0/(2\pi r^3)$ out of the brackets, we'll have:

$$\vec{E}(\vec{r},t) = -\left(\frac{\lambda_0}{2\pi}\right)\frac{Z_0 I_0 d}{4\pi r^3}\left\{\left[\cos(\omega t - k_0 r) - \left(\frac{2\pi r}{\lambda_0}\right)\sin(\omega t - k_0 r)\right](3\cos\theta\,\hat{r} - \hat{z})\right.$$

$$\left. - \left(\frac{2\pi r}{\lambda_0}\right)^2 \sin\theta\cos(\omega t - k_0 r)\,\hat{\theta}\right\}.$$

In the near-field, $r \ll \lambda_0$, we have $k_0 r \ll 1$. Therefore, we can drop terms of order $(k_0 r)^2$ and higher. Thus

$$\vec{E}(\vec{r},t) \simeq -\frac{(I_0 d/2\pi f)}{4\pi\epsilon_0 r^3}\left\{\left[\cos\omega t\cos k_0 r + \sin\omega t\sin k_0 r - (k_0 r)\sin\omega t\cos k_0 r\right.\right.$$

$$\left.\left. + (k_0 r)\cos\omega t\sin k_0 r\right](3\cos\theta\,\hat{r} - \hat{z})\right\}.$$

Now, $\sin(k_0 r) \simeq k_0 r$ and $\cos(k_0 r) = 1 - \frac{1}{2}(k_0 r)^2 + \cdots \simeq 1$. Again, eliminating terms of order $(k_0 r)^2$ and higher, we'll have:

$$\vec{E}(\vec{r},t) \cong -\frac{(I_0 d/2\pi f)}{4\pi\epsilon_0 r^3}\cos\omega t\,(3\cos\theta\,\hat{r} - \hat{z}); \quad k_0 r \ll 1$$

The charges accumulated at the top and bottom of the dipole are given by

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \Rightarrow \frac{dQ}{dt} = \pm I_0 \sin\omega t \Rightarrow Q(t) = \pm I_0\int_{-\infty}^{t}\sin\omega t'\,dt' \Rightarrow$$

$$Q(t) = \mp\frac{I_0}{\omega}\cos\omega t \Rightarrow \vec{p}(t) = Q(t)\,d\cdot\hat{z} = -\left(\frac{I_0 d}{2\pi f}\right)\cos\omega t\,\hat{z}$$

Therefore, $\vec{E}(\vec{r},t) \cong \frac{1}{4\pi\epsilon_0 r^3}\left\{3[\vec{p}(t)\cdot\hat{r}]\hat{r} - \vec{p}(t)\right\}$, $k_0 r \ll 1$.

This is the field of a static dipole $\vec{p}$, albeit with the sinusoidal variation in time.

The magnetic field of the oscillating dipole was derived (in the class) as follows:

$$\vec{B}(\vec{r},t) = \left(\frac{\lambda_0}{2\pi}\right)\frac{\mu_0 I_0 d}{4\pi r^3}\sin\theta\left\{(k_0 r)\sin(\omega t - k_0 r) + (k_0 r)^2\cos(\omega t - k_0 r)\right\}\hat{\phi}$$

$$= \left(\frac{\lambda_0}{2\pi}\right)\frac{\mu_0 I_0 d}{4\pi r^3}\sin\theta\left\{(k_0 r)\sin\omega t\cos k_0 r - (k_0 r)\cos\omega t\sin k_0 r + (k_0 r)^2\cos(\omega t - k_0 r)\right\}\hat{\phi}$$

Again, using $\sin(k_0 r) \simeq k_0 r$ and $\cos k_0 r \simeq 1$, and ignoring terms of second and higher order in $(k_0 r)$, we'll have:

$$\vec{B}(\vec{r},t) \cong \frac{\mu_0 I_0 d}{4\pi r^2}\sin\theta\sin\omega t\,\hat{\phi};\quad k_0 r \ll 1$$

The magnetic-field in the near-field region is thus proportional to the current $I(t) = I_0\sin\omega t$ in the wire, but it drops with distance as $1/r^2$ (rather than $1/r$, which would be expected from Ampère's law). Note that the near-field region does not include the immediate vicinity of the dipole, because the formulas for $\vec{E}$ and $\vec{B}$ were originally derived under the assumption that $r \gg d$. The near-field region of a dipole is therefore defined as follows: $d \ll r \ll \lambda_0$.