Problem 25)

a) $\vec{E} = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{F} = q(\vec{E} + \vec{V}\times\vec{B}) = q\left[-\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} + \vec{V}\times(\vec{\nabla}\times\vec{A})\right] \Rightarrow$$

$$\vec{F}/q = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} + (V_x\hat{x} + V_y\hat{y} + V_z\hat{z})\times\left\{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\right\} = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} + V_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{z} - V_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{y}$$

$$- V_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{z} + V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{x} + V_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{y} - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{x}$$

$$\Rightarrow \vec{F}/q = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} - \left(V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}\right)\hat{x} - \left(V_x\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_y}{\partial z}\right)\hat{y} - \left(V_x\frac{\partial A_z}{\partial x} + V_y\frac{\partial A_z}{\partial y}\right)\hat{z}$$

$$+ V_x\left(\frac{\partial A_x}{\partial y}\hat{y} + \frac{\partial A_x}{\partial z}\hat{z}\right) + V_y\left(\frac{\partial A_y}{\partial x}\hat{x} + \frac{\partial A_y}{\partial z}\hat{z}\right) + V_z\left(\frac{\partial A_z}{\partial x}\hat{x} + \frac{\partial A_z}{\partial y}\hat{y}\right)$$

The above expression can be simplified if we add and subtract the following expression: $V_x\frac{\partial A_x}{\partial x}\hat{x} + V_y\frac{\partial A_y}{\partial y}\hat{y} + V_z\frac{\partial A_z}{\partial z}\hat{z}$. We will have:

$$\vec{F}/q = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} - \left(V_x\frac{\partial \vec{A}}{\partial x} + V_y\frac{\partial \vec{A}}{\partial y} + V_z\frac{\partial \vec{A}}{\partial z}\right) + \left(V_x\vec{\nabla}A_x + V_y\vec{\nabla}A_y + V_z\vec{\nabla}A_z\right)$$

Now, $\frac{\partial \vec{A}}{\partial t} + \left(V_x\frac{\partial \vec{A}}{\partial x} + V_y\frac{\partial \vec{A}}{\partial y} + V_z\frac{\partial \vec{A}}{\partial z}\right) = \frac{\partial \vec{A}}{\partial t} + \left(\frac{\partial x}{\partial t}\frac{\partial \vec{A}}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial \vec{A}}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial \vec{A}}{\partial z}\right) = \frac{d\vec{A}}{dt}$,

where $d\vec{A}/dt$ is the total derivative of $\vec{A}(\vec{r},t)$ with respect to both $\vec{r}$ and t. In other words, if at time t the particle is at position $\vec{r}$, it experiences the vector potential $\vec{A}(\vec{r},t)$. A short time Δt later, the particle is at $\vec{r}+\Delta\vec{r} = \vec{r}+\vec{V}\Delta t$ and experiences the vector potential $\vec{A}(\vec{r}+\Delta\vec{r}, t+\Delta t)$. The total derivative of $\vec{A}$ represents this change of the field as experienced by the particle, that is, $d\vec{A}/dt = \lim_{\Delta t\to 0}\frac{\vec{A}(\vec{r}+\vec{V}\Delta t, t+\Delta t) - \vec{A}(\vec{r},t)}{\Delta t}$. The Lorentz force is thus written:

$$\vec{F}/q = -\vec{\nabla}\psi - \frac{d\vec{A}}{dt} + \vec{\nabla}(\vec{V}\cdot\vec{A})$$

← In the last term $\vec{V}$ is treated as a constant vector field. In general, $\vec{V}$ is a function of time t, but at fixed t it can be treated as a constant function of $\vec{r}$.

b) This is the familiar problem of a conducting rod moving through a uniform magnetic field. In this problem, $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\partial A_y}{\partial x}\hat{z} = \frac{A_0}{x_0}\hat{z}$ in the interval $0 < x < x_0$, and $\vec{B} = 0$ elsewhere. The $\vec{E}$-field induced along the length of the rod is, therefore, $\vec{V} \times \vec{B} = V_0 \hat{x} \times \frac{A_0}{x_0}\hat{z} = -\frac{V_0 A_0}{x_0}\hat{y}$ (when the rod crosses the region $0 < x < x_0$). The electromotive force in this region is then $\int_0^L \vec{E} \cdot d\vec{\ell} = -V_0 A_0 L / x_0$.

Using the new formulation, we have $\psi = 0$ by assumption, and $\vec{V} \cdot \vec{A} = 0$ because $\vec{V}$ is everywhere perpendicular to $\vec{A}$. Therefore, $\vec{F}/q = -d\vec{A}/dt$. Suppose at time t the rod is at x. Then at $t + \Delta t$ the rod will be at $x + V_0 \Delta t$. The $\vec{A}$-field acting on the rod thus changes from $(A_0/x_0)x\hat{y}$ to $(A_0/x_0)(x + V_0 \Delta t)\hat{y}$. The total change of $\vec{A}$ is, therefore, $(A_0/x_0)(V_0 \Delta t)\hat{y}$, yielding $\vec{F}/q = -d\vec{A}/dt = -\frac{A_0 V_0}{x_0}\hat{y}$. (This is true, of course only when $0 < x < x_0$. Outside this region $\vec{A}$ is constant in time and, consequently, $\vec{F} = 0$.) The resulting electromotive force is found as before (by integrating the force per unit charge, $\vec{F}/q$, over the length of the rod), namely, $\int_0^L (\vec{F}/q) \cdot d\ell = -V_0 A_0 L / x_0$.

In more complicated geometries, $\vec{V} \cdot \vec{A}$ may <u>not</u> vanish, and one must be careful to include the contributions of the last term, $\vec{\nabla}(\vec{V} \cdot \vec{A})$. For instance, in the case of an infinitely long solenoid, carrying a constant, uniform current around a cylindrical conductor, the $\vec{A}$-field has circular symmetry (both inside and outside the solenoid). When a rod of length L moves $\perp$ to the $\vec{B}$-field, the induced electromotive force will have some contribution from the $\vec{\nabla}(\vec{V} \cdot \vec{A})$ term.